PC2174

Tutorial 1: Vector Calculus

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1. Evaluate the integral

$$\int \left[\mathbf{a}(\dot{\mathbf{b}} \cdot \mathbf{a} + \mathbf{b} \cdot \dot{\mathbf{a}}) + \dot{\mathbf{a}}(\mathbf{b} \cdot \mathbf{a}) - 2(\dot{\mathbf{a}} \cdot \mathbf{a})\mathbf{b} - \dot{\mathbf{b}}|\mathbf{a}|^2\right] dt$$

2. The general equation of motion of a (non-relativistic) particle of mass m and charge q when it is placed in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ is

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}),$$

where $\mathbf{r}$ is the position of the particle at time t and $\dot{\mathbf{r}} = d\mathbf{r}/dt$, etc. Write this as three separate equations in terms of the Cartesian components of the vectors involved.

For the simple case of crossed uniform fields $\mathbf{E} = E\mathbf{i}$, $\mathbf{B} = B\mathbf{j}$ in which the particle starts from the origin at $t = 0$ with $\dot{\mathbf{r}} = v_0\mathbf{k}$, find the equations of motion and show the following.

(a) If $v_0 = E/B$, the particle continues its initial motion.

(b) If $v_0 = 0$, the particle follows the space curve given in terms of the parameter ξ by

$$x = \frac{mE}{B^2q}(1 - \cos\xi),\ y = 0,\ z = \frac{mE}{b^2q}(\xi - \sin\xi).$$

Interpret this curve geometrically and relate ξ to t. Show that the total distance travelled by the particle after time t is

$$\frac{2E}{B}\int_0^t \left|\sin\frac{Bqt'}{2m}\right| dt'.$$

3. Prove that for a space curve $\mathbf{r} = \mathbf{r}(s)$, where s is the arc length measured along the curve from a fixed point, the triple scalar product

$$\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3}$$

at any point on the curve has the value $\kappa^2\tau$, where κ is the curvature and τ the torsion at that point.

4. The shape of the slip road joining two motorways that cross at right angles and are at vertical heights $z = 0$ and $z = h$ can be approximated by the space curve

$$\mathbf{r} = \frac{\sqrt{2}h}{\pi} \ln\cos\left(\frac{z\pi}{2h}\right)\mathbf{i} + \frac{\sqrt{2}h}{\pi} \ln\sin\left(\frac{z\pi}{2h}\right)\mathbf{j} + z\mathbf{k}.$$

Show that at height z the radius of curvature ρ of the curve is $(2h/\pi)\csc(z\pi/h)$ and that the torsion $\tau = -1/\rho$. (To shorten the algebra, set $z = 2h\theta/\pi$ and use θ as the parameter.)

5. (a) Parametrising the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

by $x = a\cos\theta\sec\phi$, $y = b\sin\theta\sec\phi$, $z = c\tan\phi$, show that an area element on its surface is

$$dS = \sec^2\phi\left[c^2\sec^2\phi\,(b^2\cos^2\theta + a^2\sin^2\theta) + a^2b^2\tan^2\phi\right]^{1/2} d\theta d\phi.$$

(b) Use this formula to show that the area of the curved surface $x^2 + y^2 - z^2 = a^2$ between the planes $z = 0$ and $z = 2a$ is

$$\pi a^2\left(6 + \frac{1}{\sqrt{2}}\sinh^{-1} 2\sqrt{2}\right).$$