Question 1(a)

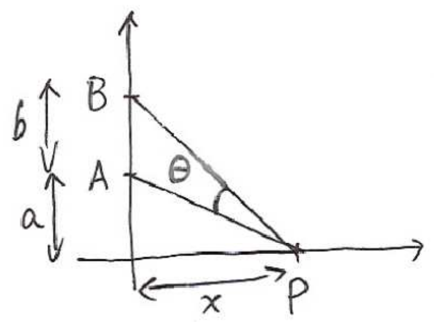
Given that $y = t + t^2 + t^5$ and $x = t^3 - t^2$, find the value of $\frac{dy}{dx}$ at the point corresponding to $t = 1$.

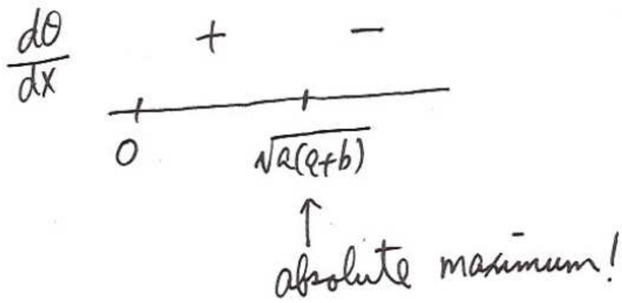
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + 2t + 5t^4}{3t^2 - 2t}$$

$$t = 1, \qquad \frac{dy}{dx} = \frac{1 + 2 + 5}{3 - 2} = 8$$

Question 1(b)

Let A be the point $(0, a)$ and B be the point $(0, a + b)$, where a and b are 2 positive constants. Let P denote a variable point $(x, 0)$, where $x > 0$. Find the value of x (in terms of a and b) that gives the largest angle $\angle APB$.



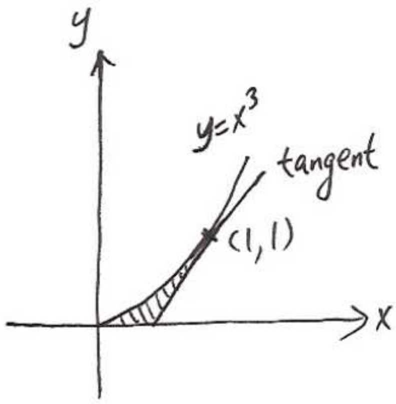


$$\theta = \tan^{-1}\frac{a+b}{x} - \tan^{-1}\frac{a}{x}$$

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{1}{1 + \left(\frac{a+b}{x}\right)^2}\frac{-(a+b)}{x^2} + \frac{1}{1 + \frac{a^2}{x^2}}\frac{a}{x^2} \\
&= -\frac{a+b}{x^2 + (a+b)^2} + \frac{a}{x^2 + a^2} \\
&= \frac{-(a+b)(x^2 + a^2) + a[x^2 + (a+b)^2]}{[x^2 + (a+b)^2](x^2 + a^2)} \\
&= \frac{b\left(\sqrt{a(a+b)} - x\right)\left(\sqrt{a(a+b)} + x\right)}{[x^2 + (a+b)^2](x^2 + a^2)} \\
&= 0
\end{aligned}$$

$$\therefore x = \sqrt{a(a+b)}$$

Question2(a)
The region R in the first quadrant of the xy-plane is bounded by the curve $y = x^3$, the x-axis and the tangent to $y = x^3$ at the point $(1,1)$. Find the area of R.



$$\frac{dy}{dx} = 3x^2$$

$$at\ (1,1), \qquad \frac{dy}{dx} = 3$$

Equation of tangent at $(1,1)$ is

$$y - 1 = 3(x-1) \quad \Rightarrow \quad x = \frac{y+2}{3}$$

∴ Area,

$$\int_0^1 \frac{y+2}{3} - y^{\frac{1}{3}}\, dy = \left[\frac{y^2}{6} + \frac{2}{3}y - \frac{3}{4}y^{\frac{4}{3}}\right]_0^1 = \frac{1}{6} + \frac{2}{3} - \frac{3}{4} = \frac{1}{12}$$

Question 2(b)
A thin rod of 2 unit length is placed on the x-axis from $x = 0$ to $x = 2$. Its density varies across the length given by the function

$$\delta(x) \begin{cases} 6 + x & 0 \le x < 1 \\ 9 - 2x & 1 \le x \le 2. \end{cases}$$

Find the x-coordinate of the center of gravity of the rod.

$$\begin{aligned}
\bar{x} &= \frac{\int_0^2 x\delta(x)\, dx}{\int_0^2 \delta(x)\, dx} \\
&= \frac{\int_0^1 x(6+x)\, dx + \int_1^2 x(9-2x)\, dx}{\int_0^1 6 + x\, dx + \int_1^2 9 - 2x\, dx} \\
&= \frac{\left[3x^2 + \frac{1}{3}x^3\right]_0^1 + \left[\frac{9}{2}x^2 - \frac{2}{3}x^3\right]_1^2}{\left[6x + \frac{1}{2}x^2\right]_0^1 + [9x - x^2]_1^2} \\
&= \frac{73}{75}
\end{aligned}$$

Question 3(a)

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(5x+2)^n.$$

$$\left|\frac{\frac{(-1)^{n+1}}{n+2}(5x+2)^{n+1}}{\frac{(-1)^n}{n+1}(5x+2)^n}\right| = \frac{n+1}{n+2}|5x+2| \quad \Rightarrow \quad |5x+2|$$

$$|5x+2| < 1 \quad \Rightarrow \quad \left|x - \frac{2}{5}\right) < \frac{1}{5}$$

$$\therefore \text{radius of convergence,} \qquad \frac{1}{5}.$$

Question 3(b)

Let $f(x) = \left|x - \frac{\pi}{2}\right|$ for all $x \in (0, \pi)$. Let

$$\sum_{n=1}^{\infty} b_n \sin nx$$

be the Fourier Since Series which represents $f(x)$. Find the value of $b_1 + b_2$.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}\left|x - \frac{\pi}{2}\right| \sin nx \, dx \\
&= \frac{2}{\pi}\left\{-\int_0^{\frac{\pi}{2}}\left(x - \frac{\pi}{2}\right)\sin nx\, dx + \int_{\frac{\pi}{2}}^{\pi}\left(x - \frac{\pi}{2}\right)\sin nx\, dx\right\} \\
&= \frac{2}{\pi}\left\{\frac{1}{n}\int_0^{\frac{\pi}{2}} x - \frac{\pi}{2}\, d(\cos nx) - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi} x - \frac{\pi}{2}\, d(\cos nx)\right\} \\
&= \frac{2}{n\pi}\left\{\left[\left(x - \frac{\pi}{2}\right)\cos nx\right]_0^{\frac{\pi}{2}} - \left[\left(x - \frac{\pi}{2}\right)\cos nx\right]_{\frac{\pi}{2}}^{\pi} - \int_0^{\frac{\pi}{2}}\cos nx\, dx + \int_{\frac{\pi}{2}}^{\pi}\cos nx\, dx\right\} \\
&= \frac{2}{n\pi}\left\{\frac{\pi}{2}[1 - (-1)^n] - \frac{2}{n}\sin\frac{n\pi}{2}\right\}
\end{aligned}$$

$$\therefore b_1 + b_2 = \frac{2}{\pi}(\pi - 2)$$

Question 4(a)
Find the distance from the point $(2,-1,4)$ to the line
$$\vec{r}(t)=\hat{\imath}+2\hat{\jmath}+7\hat{k}+t(-3\hat{\imath}+\hat{\jmath}-3\hat{k}).$$

$$\vec{a}=(2,-1,4)-(1,2,7)=(1,-3,-3)$$

$$\vec{u}=\frac{-3\hat{\imath}+\hat{\jmath}-3\hat{k}}{\sqrt{3^2+1+3^2}}=\frac{1}{\sqrt{19}}(-3\hat{\imath}+\hat{\jmath}-3\hat{k})$$

$$\vec{a}\times\vec{u}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -3 & -3\\ -3 & 1 & -3\end{vmatrix}\left(\frac{1}{\sqrt{19}}\right)=\frac{1}{\sqrt{19}}(12\hat{\imath}+12\hat{\jmath}+8\hat{k})$$

$$\therefore \text{distance}=||\vec{a}\times\vec{u}||=\frac{1}{\sqrt{19}}\sqrt{12^2+12^2+8^2}=\sqrt{\frac{352}{19}}$$

Question 4(b)
Let $f(x,y)$ be a differentiable function of 2 variables such that $f(2,1)=1506$ and $\frac{\partial f}{\partial x}(2,1)=4$. It was found that if the point Q moved from $(2,1)$ a distance 0.1 unit towards $(3,0)$, the value of f became 1505. Estimate the value of $\frac{\partial f}{\partial y}(2,1)$.

$$\frac{\partial f}{\partial y}(2,1)=a$$

unit vector from (2,1) to (3,0),
$$\vec{u}=\frac{(3,0)-(2,1)}{|(3,0)-(2,1)|}=\frac{1}{\sqrt{2}}(1,-1)$$

$$D_uf(2,1)=4\left(\frac{1}{\sqrt{2}}\right)+a\left(-\frac{1}{\sqrt{2}}\right)=\frac{4-a}{\sqrt{2}}$$

$$1505-1506\approx\frac{4-a}{\sqrt{2}}(0.1)=\frac{4-a}{10\sqrt{2}}$$

$$-10\sqrt{2}=4-a,$$

$$\therefore \frac{\partial f}{\partial y}(2,1)=a=4+10\sqrt{2}$$

Question 5(a)
Find and classify all the critical points of $f(x,y) = 4xy - 2x^2 - y^4 - 81$.

$$f_x = 0 \quad \Rightarrow \quad 4y - 4x = 0 \quad \Rightarrow \quad x = y \quad (1)$$
$$f_y = 0 \quad \Rightarrow \quad 4x - 4y^3 = 0 \quad \Rightarrow \quad x = y^3 \quad (2)$$

$$(1) = (2), \qquad y^3 = y \quad \Rightarrow \quad y = -1,0,1$$

$\therefore$ $(-1,-1), (0,0), (1,1)$ are the critical points.

critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$
$(-1,-1)$	-4	-12	4	$+$
$(0,0)$	-4	0	4	$-$
$(1,1)$	-4	-12	4	$+$

$\therefore$ $(-1,-1)$ and $(1,1)$ are local maximums. $(0,0)$ is a saddle point.

Question 5(b)
Let k be a positive constant. Evaluate

$$\iint_D x^2 e^{xy}\, dxdy$$

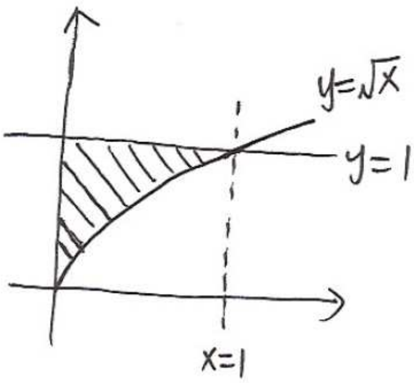
where D is the plane region given by

$$D : 0 \le x \le 2k \text{ and } 0 \le y \le \frac{1}{2k}$$

$$\begin{aligned}
\iint_D x^2 e^{xy}\, dxdy &= \int_0^{2k} \int_0^{\frac{1}{2k}} x^2 e^{xy}\, dy\, dx \\
&= \int_0^{2k} [xe^{xy}]_0^{\frac{1}{2k}}\, dx \\
&= \int_0^{2k} xe^{\frac{x}{2k}} - x\, dx \\
&= 2k \int_0^{2k} xd\left(e^{\frac{x}{2k}}\right) - \int_0^{2k} x\, dx \\
&= 2k\left\{\left[xe^{\frac{x}{2k}}\right]_0^{2k} - \int_0^{2k} e^{\frac{x}{2k}}\, dx\right\} - \left[\frac{1}{2}x^2\right]_0^{2k} \\
&= 2k\left\{2ke - 2k\left[e^{\frac{x}{2k}}\right]_0^{2k}\right\} - 2k^2 \\
&= 4k^2 - 2k^2 \\
&= 2k^2
\end{aligned}$$

Question 6(a)
Evaluate

$$\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy\, dx.$$



$$\begin{aligned}\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right) dy\, dx &= \int_0^1 \int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx\, dy \\ &= \int_0^1 y^2 \sin\left(\frac{y^3+1}{2}\right) dy \\ &= \left[-\frac{2}{3}\cos\left(\frac{y^3+1}{2}\right)\right]_0^1 \\ &= \frac{2}{3}\left(\cos\frac{1}{2} - \cos 1\right)\end{aligned}$$

Question 6(b)
Evaluate

$$\iiint_D |x|\, dxdydz$$

where D is the spherical ball of radius 2 centered at the origin.

Using spherical coordinates,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}, \qquad dxdydz = r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$\begin{aligned}\iiint_D |x|\, dx\, dy\, dz &= \int_0^{2\pi}\int_0^{\pi}\int_0^2 r\sin\phi\, |\cos\theta| r^2 \sin\phi\, dr\, d\phi\, d\theta \\ &= \int_0^2 r^3\, dr \int_0^{\pi} \sin^2\phi\, d\phi \int_0^{2\pi} |\cos\theta|\, d\theta \\ &= 4\left[\frac{1}{4}r^4\right]_0^2 \int_0^{\pi} \frac{1-\cos 2\phi}{2} d\phi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \\ &= 4 \times \frac{\pi}{2} \times 4 \\ &= 8\pi\end{aligned}$$

Question 7(a)
A force given by the vector field $\vec{F} = (y+z)\hat{\imath} + (x+2yz)\hat{\jmath} + (x+y^2)\hat{k}$ moves a particle from point $P(0,0,0)$ to point $Q(1,2,3)$. Find the work done by $\vec{F}$.

$$\overrightarrow{PQ}, \qquad \vec{r}(t) = (t, 2t, 3t), \qquad 0 \le t \le 1$$

∴ Work done,

$$\int_0^1 F[\vec{r}(t)] \cdot \vec{r}'(t)\, dt = \int_0^1 36t^2 + 10t\, dt = 17$$

<u>Question 7(b)</u>
Evaluate the line integral

$$\int_C \left(\ln\sqrt{1+x^2} - y^3\right) dx + \left(x^3 + \sqrt{1-\sin^3 y}\right) dy$$

where C is the boundary with positive orientation of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Let D be the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Applying Green's Theorem, we have

$$\begin{aligned}
\int_C \left(\ln\sqrt{1+x^2} - y^3\right) dx + \left(x^3 + \sqrt{1-\sin^3 y}\right) dy &= \iint_R 3x^2 + 3y^2\, dx\, dy \\
&= 3\int_0^{2\pi}\int_1^2 r^3\, dr\, d\theta \\
&= 6\pi\left[\frac{1}{4}r^4\right]_1^2 \\
&= \frac{45\pi}{2}
\end{aligned}$$

<u>Question 8(a)</u>
Evaluate $\iint_S F \cdot dS$ where $F = y^2\hat{\imath} + x^2\hat{\jmath} + z\hat{k}$ and S is the portion of the plane $x + y + z - 1 = 0$ in the first octant. The orientation of S is given by the upward normal vector.

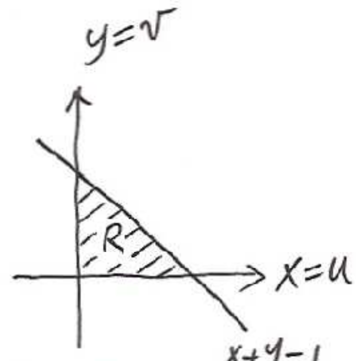
$$z = 1 - x - y$$

Parametric representation of S,

$$\vec{r}(u, v) = u\hat{\imath} + v\hat{\jmath} + (1 - u - v)\hat{k}$$

$$\vec{r}_u = \hat{\imath} - \hat{k}, \qquad \vec{r}_v = \hat{\jmath} - \hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + \hat{k}$$



Projection of S onto the xy-plane

$$\iint_S F \cdot dS$$

$$
\begin{aligned}
&= \iint_R F \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv \\
&= \iint_R v^2 + u^2 + 1 - u - v\, du\, dv \\
&= \int_0^1 \int_0^{1-v} v^2 + u^2 + 1 - u - v\, du\, dv \\
&= \int_0^1 \left[v^2 u + \frac{1}{3}u^3 + u - \frac{1}{2}u^2 - vu \right]_0^{1-v} dv \\
&= \int_0^1 2v^2 - v^3 - 2v + \frac{1}{3}(1-v)^3 + 1 - \frac{1}{2}(1-v)^2\, dv \\
&= \left[\frac{2}{3}v^3 - \frac{1}{4}v^4 - v^2 + \frac{1}{12}(1-v)^4 + v + \frac{1}{6}(1-v)^3 \right]_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

Question 8(b)

Using the method of separation of variables, solve the partial differential equation

$$\boldsymbol{x u_x - y u_x = 0},$$

where $x > 0$ and $y > 0$.

$$u = XY$$

$$xX'Y - yXY' = 0 \quad \Rightarrow \quad xX'Y = yXY' \quad \Rightarrow \quad x\frac{X'}{X} = y\frac{Y'}{Y} = c$$

$$\frac{X'}{X} = \frac{c}{x}, \qquad \frac{Y'}{Y} = \frac{c}{y}$$

$$
\begin{aligned}
\ln|X| &= c\ln|x| + a, \qquad X = k_1 x^c \\
\ln|Y| &= c\ln|y| + b, \qquad Y = k_2 y^c
\end{aligned}
$$

$$\therefore u = XY = k(xy)^c$$