Question 1(a)

Find the slope of the tangent to the curve $x = t - \sin t,\ y = 1 - \cos t$, at the point corresponding to $t = \frac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t}$$

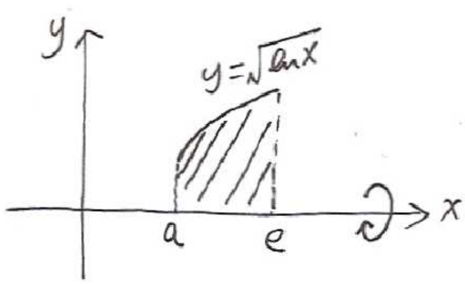
$$t = \frac{\pi}{3}, \qquad \frac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \sqrt{3}$$

Question 1(b)

Let a be a positive constant and $1 < a < e$. Let R denote the finite region in the 1st quadrant bounded by the curve $y = \sqrt{\ln x}$, the x-axis, the line $x = a$ and the line $x = e$. Find the exact value of the volume of the solid formed by revolving R one complete round about the x-axis.

Volume,

$$\int_a^e \pi y^2\, dx = \pi \int_a^e \ln x\, dx = \pi[x \ln x - x]_a^e = \pi(a - a \ln a)$$



Question 2(a)

Let

$$f(x) = \frac{x^2 + 1}{x + 1}$$

and let

$$\sum_{n=0}^{\infty} c_n (x+3)^n$$

be the Taylor series for f at $x = -3$. Find the exact value of $c_0 + c_1 + c_{101}$.

$$f(x) = \frac{x^2+1}{x+1}$$

$$= x - 1 + \frac{2}{x+1}$$

$$= (x+3) - 4 + \frac{2}{(x+3)-2}$$

$$= -4 + (x+3) - \frac{1}{1 - \left(\frac{x+3}{2}\right)}$$

$$= -4 + (x+3) - \sum_{n=0}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$= -5 + \frac{1}{2}(x+3) - \sum_{n=2}^{\infty} \frac{1}{2^n}(x+3)^n$$

$$\therefore c_0 + c_1 + c_{101} = -5 + \frac{1}{2} - \frac{1}{2^{101}} = -\frac{9}{2} - \frac{1}{2^{101}}$$

Question 2(b)

A car is moving with speed $20\mathrm{ms}^{-1}$ and acceleration $\alpha\mathrm{ms}^{-2}$ at a given instant. The car is observed to have moved a distance of $29\mathrm{m}$ in the next second. Using a 2nd degree Taylor polynomial, estimate the value of α.

We may assume that the car is at the origin with $t = 0$ when $v = 20\mathrm{ms}^{-1}$ and acceleration $= \alpha\mathrm{ms}^{-2}$.

x =distance from origin at time t.

$$\frac{dx}{dt}(0) = 20, \qquad \frac{d^2x}{dt^2}(0) = \alpha$$

$$x \approx 0 + 20t + \frac{\alpha}{2!}t^2$$

$$x = 29, t = 1 \quad \Rightarrow \quad 29 = 20 + \frac{\alpha}{2}$$

$$\therefore \alpha = 18$$

Question 3(a)

Let

$$f(x) = x^2\sqrt{\pi^2 - x^2}, \qquad -\pi \le x \le \pi,$$

and $f(x + 2\pi) = f(x)$ for all x. Let

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the exact value of

$$b_2 + b_3 + \sum_{n=1}^{\infty} a_n.$$

f is even, $b_n = 0$ for all n.

$$x = 0 \quad \Rightarrow \quad a_0 + \sum_{n=1}^{\infty} a_n = f(0) = 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} x^2\sqrt{\pi^2 - x^2}\,dx$$

$$x = \pi \sin\theta\,, \qquad dx = \pi\cos\theta\,d\theta$$

$$a_0 = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}(\pi^2\sin^2\theta)(\pi\cos\theta)^2\,d\theta = \frac{\pi^3}{4}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,d\theta = \frac{\pi^3}{8}\int_0^{\frac{\pi}{2}} 1 - \cos 2\theta\,d\theta = \frac{\pi^4}{16}$$

$$\therefore\, b_2 b_3 + \sum_{n=1}^{\infty} a_n = -a_0 = -\frac{\pi^4}{8}$$

Question 3(b)

Let $f(x) = x - 1,\;\; 0 < x < 1$. Let

$$\sum_{n=1}^{\infty} b_n \sin n\pi x$$

be the sine Fourier half range expansion for $f(x)$. Find the exact value of b_{2008}.

$$\begin{aligned}
b_{2008} &= \frac{2}{1}\int_0^1 f(x)\sin 2008\pi x\,dx \\
&= 2\left(\int_0^1 x\sin 2008\pi x\,dx - \int_0^1 \sin 2008\pi x\,dx\right) \\
&= 2\left(-\frac{1}{2008\pi}[x\cos 2008\pi x]_0^1 - \int_0^1 \cos 2008\pi x\,dx + \frac{1}{2008\pi}[\cos 2008\pi x]_0^1\right) \\
&= -\frac{1}{1004\pi}\left(1 - \frac{1}{2008\pi}[\sin 2008\pi x]_0^1\right)
\end{aligned}$$

$$= -\frac{1}{1004\pi}$$

Question 4(a)

Let S be the plane which passes through the points $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$. Find the shortest distance from the point $(-1,-2,-3)$ to S.

$$S : \frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1 \quad \Rightarrow \quad 6x + 3y + 2z = 6$$

∴ shortest distance,

$$\frac{|6(-1) + 3(-2) + 2(-3) - 6|}{\sqrt{6^2 + 3^2 + 2^2}} = \frac{24}{7}$$

Question 4(b)

Let $\vec{A}$ and $\vec{B}$ be 2 non-zero constant vectors and $|\vec{B}| = 2$. If

$$\lim_{x\to\infty}\left(\left|x\vec{A} + \vec{B}\right| - \left|x\vec{A}\right|\right) = -\frac{1}{5},$$

find the exact value of $\cos\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

$$\begin{aligned}
\lim_{x\to\infty}\left(\left|x\vec{A} + \vec{B}\right| - \left|x\vec{A}\right|\right) &= \lim_{x\to\infty}\frac{\left|x\vec{A} + \vec{B}\right|^2 - \left|x\vec{A}\right|^2}{\left|x\vec{A} + \vec{B}\right| + \left|x\vec{A}\right|} \\
&= \lim_{x\to\infty}\frac{\left(x\vec{A} + \vec{B}\right)\cdot\left(x\vec{A} + \vec{B}\right) - \left|x\vec{A}\right|^2}{\left|x\vec{A} + \vec{B}\right| + \left|x\vec{A}\right|} \\
&= \lim_{x\to\infty}\frac{2x\vec{A}\cdot\vec{B} + \left|\vec{B}\right|^2}{\left|x\vec{A} + \vec{B}\right| + \left|x\vec{A}\right|} \\
&= \lim_{x\to\infty}\frac{2\vec{A}\cdot\vec{B} + \frac{\left|\vec{B}\right|^2}{x}}{\left|\vec{A} + \frac{\vec{B}}{x}\right| + \left|\vec{A}\right|} \\
&= \frac{\vec{A}\cdot\vec{B}}{\left|\vec{A}\right|} \\
&= \left|\vec{B}\right|\cos\theta \\
&= 2\cos\theta
\end{aligned}$$

$$= -\frac{1}{5}$$

$$\therefore \cos\theta = -\frac{1}{10}$$

Question 5(a)

Let $f(x,y,z)$ be a differentiable function of 3 variables, P be a point in space and $f(P) = 1$. It is known that the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}$ at P are equal to $-\sqrt{3}, -\frac{\sqrt{3}}{4}, -\frac{1}{\sqrt{12}}$ respectively. Suppose P moves 0.1 unit in the direction of the vector $\vec{u} = \hat{\imath} - \hat{\jmath} - \hat{k}$ to the point Q. Estimate the value of $f(Q)$.

$$\vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\nabla f(P) = \begin{pmatrix} -\sqrt{3} \\ -\frac{\sqrt{3}}{4} \\ -\frac{1}{\sqrt{12}} \end{pmatrix}, \qquad D_{\vec{v}} f(P) = \nabla f(P) \cdot \vec{v} = -1 + \frac{1}{4} + \frac{1}{6} = -\frac{7}{12}$$

$$f(Q) - f(P) \approx D_{\vec{v}} f(P) \times 0.1 = -\frac{7}{120}$$

$$\therefore f(Q) = 1 - \frac{7}{120} = \frac{113}{120}$$

Question 5(b)

Let n be a fixed positive integer and $n \geq 2$. FInd, if any, the local maximum points, the local minimum points and the saddle points of the function $f(x,y) = \ln(x^n y) - xy - (n-1)x$, where $x > 0$ and $y > 0$.

$$f_x = \frac{n}{x} - y - n + 1 = 0, \qquad f_y = \frac{1}{y} - x = 0$$

$$y = \frac{1}{x}, \qquad x = 1, \qquad (1,1) \text{ is the only critical point.}$$

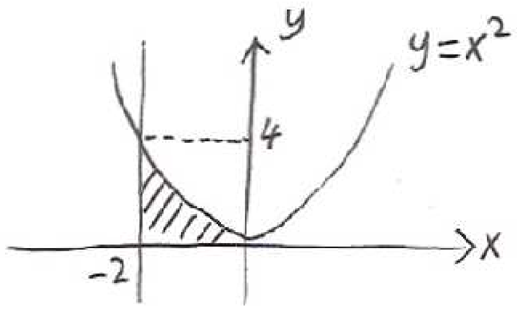
$$f_{xx} = -\frac{n}{x^2}, \qquad f_{xy} = -1, \qquad f_{yy} = -\frac{1}{y^2}$$

at $(1,1)$, $f_{xx}f_{yy} - f_{xy}^2 = n - 1 > 0$, $f_{xx} = -n < 0$

$\therefore$ $(1,1)$ is a local maximum.

Question 6(a)
Find the exact value of the integral

$$\int_0^4 \int_{-2}^{-\sqrt{y}} e^{x^3}\, dx\, dy.$$



$$\therefore \int_0^4 \int_{-2}^{-\sqrt{y}} e^{x^3}\, dx\, dy = \int_{-2}^{0} \int_0^{x^2} e^{x^3}\, dy\, dx = \int_{-2}^{0} x^2 e^{x^3}\, dx = \left[\frac{1}{3} e^{x^3}\right]_{-2}^{0} = \frac{1}{3} - \frac{1}{3} e^{-8}$$

Question 6(b)
Find the exact value of the surface area of the part of the surface $z = 2 - x^2 - y^2$ which lies above the xy-plane.

$z_x = -2x, \qquad z_y = -2y$

$\therefore$ the surface area,

$$\iint_{x^2+y^2\le 2} \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy = \iint_{x^2+y^2\le 2} \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr\, d\theta$$

$$= 2\pi \left[\frac{1}{12}(1 + 4r^2)^{\frac{3}{2}}\right]_0^{\sqrt{2}}$$

$$= \frac{13\pi}{3}$$

<u>Question 7(a)</u>

Let a be a positive odd integer. Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F} = \frac{y}{a}\hat{\imath} - \frac{x}{a}\hat{\jmath} + \frac{2}{a}\hat{k}$ and C is the helix $\vec{r}(t) = \cos t\hat{\imath} + \sin t\hat{\jmath} + t\hat{k}$ from $(1,0,0)$ to $(-1,0,a\pi)$.

$$d\vec{r} = \left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}\right)dt$$

$$\vec{F}[\vec{r}(t)] = \frac{\sin t}{a}\hat{\imath} - \frac{\cos t}{a}\hat{\jmath} + \frac{2}{a}\hat{k}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^{a\pi} -\frac{\sin^2 t}{a} - \frac{\cos^2 t}{a} + \frac{2}{a}dt = \int_0^{a\pi} \frac{1}{a}dt = \pi$$

<u>Question 7(b)</u>

Find the exact value of the surface integral

$$\iint_S x + y\, d\vec{S},$$

where $\vec{S}$ is the surface defined parametrically by

$$\vec{r}(u,v) = u\hat{\imath} + 3\cos v\hat{\jmath} + 3\sin v\hat{k}, \qquad \left(0 \le u \le 4, \qquad 0 \le v \le \frac{\pi}{2}\right).$$

$$\vec{r}_u = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \vec{r}_v = \begin{pmatrix}0\\-3\sin v\\3\cos v\end{pmatrix}, \qquad \vec{r}_u \times \vec{r}_v = \begin{pmatrix}0\\-3\cos v\\-3\sin v\end{pmatrix}$$

$$|\vec{r}_u \times \vec{r}_v| = 3$$

$$\begin{aligned}\therefore \iint_S x + y\, d\vec{S} &= \int_0^{\frac{\pi}{2}}\int_0^4 3(u + 3\cos v)\, du\, dv \\ &= 3\int_0^{\frac{\pi}{2}} 8 + 12\cos v\, dv \\ &= 3[8v + 12\sin v]_0^{\frac{\pi}{2}} \\ &= 12\pi + 36\end{aligned}$$

<u>Question 8(a)</u>

Use Stokes' Theorem to find the exact value of the line integral

$$\oint_C \left(y\,dx - \frac{1}{2}z^2\,dy + \frac{1}{2}x^2\,dz \right),$$

where C is the curve of intersection of the plane $y + z = 0$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 12$, oriented counterclockwise as seen from above.

Let S be the region on $y + z = 0$ and bounded by C.

$y + z = 0, \qquad 3x^2 + 2y^2 + z^2 = 12 \quad \Rightarrow \quad x^2 + y^2 = 4$

$S : \vec{r}(u,v) = u\hat{\imath} + v\hat{\jmath} - v\hat{k}, \qquad u^2 + v^2 \le 4$

$$\vec{r}_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{r}_v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \vec{r}_u \times \vec{r}_v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$d\vec{S} = (\hat{\jmath} + \hat{k})du\,dv$

$\vec{r}_u \times \vec{r}_v$ points upwards, the orientation of S agrees with the orientation of C.

$$\vec{F} = y\hat{\imath} - \frac{1}{2}z^2\hat{\jmath} + \frac{1}{2}x^2\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -\dfrac{1}{2}z^2 & \dfrac{1}{2}x^2 \end{vmatrix} = z\hat{\imath} - x\hat{\jmath} - \hat{k}$$

$$\begin{aligned} \therefore \oint_C \vec{F} \cdot d\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \\ &= \iint_{u^2+v^2\le 4} -u - 1\,du\,dv \\ &= \int_0^2 \int_0^{2\pi} r(-r\cos\theta - 1)\,d\theta\,dr \\ &= \int_0^2 -2\pi r\,dr \\ &= [-\pi r^2]_0^2 \\ &= -4\pi \end{aligned}$$

Question 8(b)

Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$\boldsymbol{u_x + u_y = (x-y)u,}$$

given that $u(0,0) = u(0,2) = 1$.

$u = X(x)Y(y)$

$X'Y + XY' = (x-y)XY \quad \Rightarrow \quad \dfrac{X'}{X} + \dfrac{Y'}{Y} = x - y$

$\dfrac{X'}{X} = x + k, \qquad \dfrac{Y'}{Y} = -y - k$

$\ln|X| = \dfrac{1}{2}x^2 + kx + c_1, \qquad \ln|Y| = -\dfrac{1}{2}y^2 - ky + c_2$

$u = XY = ce^{\frac{1}{2}(x^2-y^2)+k(x-y)}$

$u(0,0) = u(0,2) = 1 \quad \Rightarrow \quad c = 1, \qquad k = -1$

$\therefore u = e^{\frac{1}{2}(x^2-y^2)-x+y}$