Question 1(a)
Given that $y^2 - 4x = 4 - 4y$. Find the value of $\frac{dy}{dx}$ at the point $(2,2)$.

$$2yy' - 4 = -4y'$$

$$(2y + 4)y' = 4$$

$$y' = \frac{4}{2y + 4}$$

$$\therefore y'(2,2) = \frac{4}{2(2) + 4} = \frac{1}{2}$$

Question 1(b)
Let $f(x) = (\sin x)^{\sin x}$ for all $x \in \left(0, \frac{\pi}{2}\right)$. Given that f has a critical point at $c \in \left(0, \frac{\pi}{2}\right)$. Find the value of $\sin c$.

$$\ln f = (\sin x)\ln(\sin x)$$

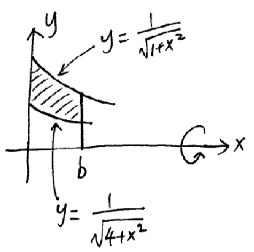
$$\frac{1}{f}f' = (\cos x)\ln(\sin x) + \frac{\sin x}{\sin x}\cos x = \cos x\,[\ln(\sin x) + 1]$$

$$f'(c) = 0 \text{ for } c \in \left(0, \frac{\pi}{2}\right)$$

$$\ln(\sin c) + 1 = 0$$

$$\therefore \sin c = \frac{1}{e}$$

Question 2(a)
The region bounded by the graphs of $y = \frac{1}{\sqrt{1+x^2}}$, $y = \frac{1}{\sqrt{4+x^2}}$, $x = 0$ and $x = b$ where b denotes a positive constant is rotated about the x-axis to generate a solid of revolution. Let $V(b)$ denote the volume of this solid of revolution. Find the value of $\lim_{b\to\infty} V(b)$.



$$V(b) = \pi\int_0^b \left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{1}{\sqrt{4+x^2}}\right)^2 dx$$
$$= \pi\int_0^b \frac{1}{1+x^2} - \frac{1}{4+x^2}\,dx$$

$$= \pi\left[\tan^{-1} x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^b$$
$$= \pi\left(\tan^{-1} b - \frac{1}{2}\tan^{-1}\frac{b}{2}\right)$$

$$\lim_{b\to\infty} V(b) = \pi\left[\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right)\right] = \frac{\pi^2}{4}$$

Question 2(b)
Find the value of

$$\frac{\int_{-\frac{\pi}{2}}^{0} \cos^{10} x\, dx}{\int_{-\frac{\pi}{2}}^{0} \cos^{8} x\, dx}.$$

$$\int_{-\frac{\pi}{2}}^{0} \cos^{10} x\, dx = \int_{-\frac{\pi}{2}}^{0} \cos^{9} x\, d(\sin x)$$
$$= [\cos^9 x \sin x]_{-\frac{\pi}{2}}^{0} + \int_{-\frac{\pi}{2}}^{0} 9\cos^9 x \sin^2 x\, dx$$
$$= 9\int_{-\frac{\pi}{2}}^{0} \cos^8 x\,(1-\cos^2 x)\, dx$$
$$= 9\int_{-\frac{\pi}{2}}^{0} \cos^8 x\, dx - 9\int_{-\frac{\pi}{2}}^{0} \cos^{10} x\, dx$$

$$10\int_{-\frac{\pi}{2}}^{0} \cos^{10} x\, dx = 9\int_{-\frac{\pi}{2}}^{0} \cos^{8} x\, dx$$

$$\frac{\int_{-\frac{\pi}{2}}^{0} \cos^{10} x\, dx}{\int_{-\frac{\pi}{2}}^{0} \cos^{8} x\, dx} = \frac{9}{10}$$

Question 3(a)
Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8^n + (-9)^n}{n+1}(x+2)^{2n}.$$

$$\lim_{n\to\infty} \frac{\frac{8^{n+1} + (-9)^{n+1}}{n+2}(x+2)^{2n+2}}{\frac{8^n + (-9)^n}{n+1}(x+2)^{2n}} = \lim_{n\to\infty} \frac{n+1}{n+2}\left|\frac{8\left(\frac{8}{9}\right)^n + 9(-1)^n}{\left(\frac{8}{9}\right)^n + (-1)^n}\right| |x+2|^2 = 9|x+2|^2$$

$$9|x+2|^2 < 1, \qquad \Rightarrow |x+2| < \frac{1}{3}, \qquad \therefore |x-(-2)| < \frac{1}{3}$$

Question 3(b)

Let $f(x) = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ where – 12 $\le x \le \frac{1}{2}$. Find the value of $f^{(2005)}(0)$. Give your answer in terms of factorials.

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\left(\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right)$$
$$= \frac{2}{(1-x)^2+(1+x)^2}$$
$$= \frac{1}{1+x^2}$$
$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int_0^x f'(t)\,dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\,dt$$

$$f(x) - f(0) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \qquad \left[f(0) = \tan^{-1}1 = \frac{\pi}{4}\right]$$

$$\therefore f^{(2005)}(0) = \frac{2005!\,(-1)^{\frac{2005-1}{2}}}{2005} = 2004!$$

Question 4(a)

Let $f(x) = \cos\frac{x}{2}$ for all $x \in (0,\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

be the Fourier Cosine Series which represents $f(x)$. Find the value of $a_0 + a_1$. Give your answer in terms of π.

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\cos\frac{x}{2}\,dx = \left[\frac{2}{\pi}\sin\frac{x}{2}\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi}\cos\frac{x}{2}\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos\frac{x}{2}+\cos\frac{3x}{2}\,dx = \frac{1}{\pi}\left[2\sin\frac{x}{2}+\frac{2}{3}\sin\frac{3x}{2}\right]_0^{\pi} = \frac{4}{3\pi}$$

$$\therefore a_0 + a_1 = \frac{2}{\pi} + \frac{4}{3\pi} = \frac{10}{3\pi}$$

Question 4(b)
Let $f(x) = 2x + 1$ for all $x \in (-\pi, \pi)$ and $f(x) = f(x + 2\pi)$. Let

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier Series which represents $f(x)$. Find the value of $a_0 + a_5 + b_5$.

The function $g(x) = x$ is an odd function on $(-\pi, \pi)$.

$$g(x) \approx \sum_{n=1}^{\infty} c_n \sin nx$$

$$\begin{aligned} c_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \\ &= -\frac{2}{n\pi} \int_{0}^{\pi} x \, d(\cos nx) \\ &= -\frac{2}{n\pi} [x \cos nx]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \cos nx \, dx \\ &= -\frac{2}{n} (\cos n\pi) \\ &= (-1)^{n+1} \frac{2}{n} \end{aligned}$$

$$f(x) = 2x + 1 = 2g(x) + 1 \approx 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx$$

$$\therefore a_0 + a_5 + b_5 = 1 + 0 + (-1)^6 \frac{4}{5} = \frac{9}{5}$$

Question 5(a)
The number of bacteria in a certain bacterium culture is 1000 at a certain initial time. 2 hours after the initial time there are 1200 of them. Assuming constant birth and death rates per capita, how many bacteria will we have 6 hours after the initial time?

$$\frac{dN}{dt} = (B - D)N \quad \Rightarrow \quad N = N_0 e^{(B-D)t}$$

$$N_0 = 1000 \quad \Rightarrow \quad N = 1000 e^{(B-D)t}$$

$$t = 2, N = 1200 \quad \Rightarrow \quad 1200 = 1000 e^{2(B-D)}$$

$$B - D = \frac{1}{2}\ln\frac{12}{10}$$

At $t = 6,$ we have

$$N = 1000e^{\left(\frac{1}{2}\ln\frac{12}{10}\right)6} = 1728$$

Question 5(b)

Solve the differential equation

$$(\boldsymbol{y^2 - x^2})\frac{\boldsymbol{dy}}{\boldsymbol{dx}} + \boldsymbol{2xy = 0}$$

with $x > 0$ and the initial condition $y = 1$ when $x = \sqrt{2}$.

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx}x + v = \frac{2xy}{x^2 - y^2} = \frac{2x^2v}{x^2 - v^2x^2} = \frac{2v}{1 - v^2}$$

$$\frac{dv}{dx}x = \frac{2v}{1 - v^2} - v = \frac{v + v^3}{1 - v^2} = \frac{v(1 + v^2)}{1 - v^2}$$

$$\frac{1 - v^2}{v(1 + v^2)}dv = \frac{1}{v} - \frac{2v}{1 + v^2}\,dv = \frac{1}{x}\,dx$$

$$\ln|v| - \ln|1 + v^2| = \ln x + \ln c_1$$

$$\frac{v}{1 + v^2} = cx \quad \Rightarrow \quad \frac{y}{x\left[1 + \left(\frac{y}{x}\right)^2\right]} = cx \quad \Rightarrow \quad \frac{y}{x^2 + y^2} = c$$

$$y = c(x^2 + y^2)$$

$$x = \sqrt{2}, y = 1 \quad \Rightarrow \quad c = \frac{1}{3}$$

$$\therefore x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

Question 6(a)

A water tank has a capacity of $120\,l$. Initially the tank contains $90l$ of salt solution with a concentration of $1\,\mathrm{g}l^{-1}$. A tap is then turned on and a salt solution with a concentration of $2\,\mathrm{g}l^{-1}$ enters the tank at a rate of $4\,l\mathrm{min}^{-1}$. At the same time when the tap is turned on, a drain is also turned on and the well-stirred mixture flows out of the tank at a rate of $3\,l\mathrm{min}^{-1}$. How much salt in grams (round off to the nearest integer) does the tank contain at the moment when it is full?

Suppose the tank contains x grams of salt at t minutes pass the initial time. Note that at this time, the tank contains $90 + t(4-3) = 90 + t$ litres of salt solution.

$$\frac{dx}{dt} = 8 - \frac{3x}{90+t} \quad \Rightarrow \quad \frac{dx}{dt} + \frac{3}{90+t}x = 8$$

$$e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3$$

$$\frac{d}{dt}[x(90+t)^3] = 8(90+t)^3$$

$$x(90+t)^3 = 2(90+t)^4 + c$$

$$x = 2(90+t) + \frac{c}{(90+t)^3}$$

$$t = 0, x = 90 \quad \Rightarrow \quad c = -90^4 \quad x = \frac{2(90+t)^4 - 90^4}{(90+t)^3}$$

$$\text{the tank is full when } 90 + \text{t} = 120 \quad \Rightarrow \quad \text{t} = 30$$

$$\therefore x = \frac{2(120)^4 - 90^4}{(120)^3} = 202\frac{1}{32} \approx 202$$

<u>Question 6(b)</u>
Solve the differential equation

$$\boldsymbol{y'' - y' - 2y = 0}$$

with the initial conditions that $\boldsymbol{y = 1}$ and $\boldsymbol{y' = 5}$ when $\boldsymbol{x = 0}$.

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = -1, 2$$

$$y = Ae^{-x} + Be^{2x}$$
$$y' = -Ae^{-x} + 2Be^{2x}$$

$$y(0) = 1, y'(0) = 5 \quad \Rightarrow \quad A + B = 1, \qquad -A + 2B = 5$$
$$A = -1, \qquad B = 2$$

$$\therefore y = -e^{-x} + 2e^{2x}$$

Question 7(a)
Solve the differential equation

$$y'' - 5y' + 6y = 4e^{2x}$$

with the initial conditions that $y = 0$ and $y' = 1$ when $x = 0$.

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad (\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 2,3$$

$$y = (Ax + B)e^{2x}$$
$$y' = Ae^{2x} + 2(Ax + B)e^{2x}$$
$$y'' = 4e^{2x} + 4(Ax + B)e^{2x}$$

$$4e^{2x} + 4(Ax + B)e^{2x} - 5[Ae^{2x} + 2(Ax + B)e^{2x}] + 6(Ax + B)e^{2x} = -Ae^{2x}$$

$$A = -4$$

$$y = Ce^{2x} + De^{3x} - 4xe^{2x}$$
$$y' = (2 - 4)Ce^{2x} + 3De^{3x} - 8xe^{2x}$$

$$y(0) = 0, y'(0) = 1 \quad \Rightarrow \quad C + D = 0, \qquad 2C + 3D = 5$$

$$C = -5, \qquad D = 5$$

$$\therefore y = 5e^{3x} - 5e^{2x} - 4xe^{2x}$$

Question 7(b)
Solve the differential equation

$$y'' + y = \tan^2 x$$

with the initial conditions that $y = 1$ and $y' = 1$ when $x = 0$.

$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i$$

$$f(x) = \tan^2 x\,, \qquad y_1 = \cos x\,, \qquad y_2 = \sin x$$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix} = 1$$

$$u = -\int \frac{y_2 f(x)}{w(y_1, y_2)} dx = -\int \frac{\sin^3 x}{\cos^2 x} dx = \int \frac{1 - \cos^2 x}{\cos^2 x} d(\cos x) = -\frac{1}{\cos x} - \cos x$$
$$v = \int \frac{y_1 f(x)}{w(y_1, y_2)} dx = -\int \frac{\sin^2 x}{\cos x} dx = \int \sec x - \cos x\, dx = \ln|\sec x + \tan x| - \sin x$$

$$y = A\cos x + B\sin x + uy_1 + vy_2 = A\cos x + B\sin x - 2 + \sin x \ln|\sec x + \tan x|$$

$$y(0) = y'(0) = 1 \quad \Rightarrow \quad A = 3, \qquad B = 1$$

$$\therefore y = 3\cos x + \sin x - 2 + \sin x \ln|\sec x + \tan x|$$

Question 8(a)
Find the Laplace transform $\boldsymbol{L[(\sin t - \cos t)^3]}$.

$$\begin{aligned} L[(\sin t - \cos t)^3] &= L(\sin^2 t - 2\sin t\cos t + \cos^2 t) \\ &= L(1 - \sin 2t) \\ &= \frac{1}{s} - \frac{2}{s^2+4} \\ &= \frac{s^2 - 2s + 4}{s(s^2+4)} \end{aligned}$$

Question 8(b)
Find the inverse Laplace transform

$$\boldsymbol{L^{-1}\left(e^{-s}\frac{s+3}{s^2+4s+4}\right)}.$$

$$F(s) = \frac{s+3}{s^2+4s+4} = \frac{1}{s+2} + \frac{1}{(s+2)^2}$$

$$f(t) = L^{-1}[F(s)] = e^{-2t} + te^{-2t} = (1+t)e^{-2t}$$

$$\therefore L^{-1}\left(e^{-s}\frac{s+3}{s^2+4s+4}\right) = L^{-1}[e^{-s}F(s)] = f(t-1)U(t-1) = U(t-1)te^{-2t}$$