Question 1(a)
Find the slope of the tangent to the curve $y = (35x - 69)^{43}$ when $x = 2$.

$y' = 43(35x - 69)^{42}(35)$

$x = 2$

$\therefore y' = 43(70 - 69)^{42}(35) = 1505$

Question 1(b)
Let $f(x) = ax^3 + bx^2$ be a function defined on $(-\infty, \infty)$, where a and b are non-zero constants. Given that f has a point of inflection at $(1, 2)$, find the value of the product ab.

$f(x) = ax^3 + bx^2$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b = 2(3ax + b)$

$f(1) = 2 \quad \Rightarrow \quad a + b = 2$

f'' changes sign at $x = 1 \quad \Rightarrow \quad 3a + b = 0$

$a = -1, \qquad b = 3$

$\therefore ab = -3$

Question 2(a)
Let

$$f(x) = \frac{23 - 4x}{7 - 2x}$$

and let

$$\sum_{n=0}^{\infty} c_n (x-2)^n$$

be the Taylor series for f at $x = 2$. Find the exact value of $c_0 + c_{2009}$.

$$f(x) = \frac{14 - 4x + 9}{7 - 2x}$$

$$= 2 + \frac{9}{7 - 2(x-2) - 4}$$

$$= 2 + \frac{3}{1 - \frac{2}{3}(x-2)}$$

$$= 2 + 3\sum_{n=0}^{\infty} \frac{2^n}{3^n}(x-2)^n$$

$$= 5 + \sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}}(x-2)^n$$

$$\therefore c_0 + c_{2009} = 5 + \frac{2^{2009}}{3^{2008}}$$

Question 2(b)

Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$2u_{xy} = [\sin(x+y) + \sin(x-y)]u,$$

given that $u(0,0) = 1$ and $u(\pi,\pi) = e^2$.

$$u = X(x)Y(y)$$

$$2X'Y' = 2XY \sin x \cos y$$

$$\frac{X'}{X \sin x} = \frac{Y \cos y}{Y'} = k$$

$$\frac{X'}{X} = k \sin x \quad \Rightarrow \quad \ln|x| = -k\cos x \quad \Rightarrow \quad X = Ae^{-k\cos x}$$

$$\frac{Y'}{Y} = \frac{1}{k}\cos y \quad \Rightarrow \quad \ln|y| = \frac{1}{k}\sin y \quad \Rightarrow \quad Y = Be^{\frac{1}{k}\sin y}$$

$$u = ce^{-k\cos x + \frac{1}{k}\sin y}$$

$$u(0,0) = 1, u(\pi,\pi) = e^2 \quad \Rightarrow \quad k = 1, c = e$$

$$\therefore u = e^{1 - \cos x + \sin y}$$

Question 3(a)

Let

$$\boldsymbol{f(x) = x^2, \qquad -\pi \le x \le \pi,}$$

and $f(x+2\pi) = f(x)$ for all x. Let

$$\boldsymbol{a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

be the Fourier Series which represents $f(x)$. Find the exact value of

$$\boldsymbol{a_{2010} + b_{2010}}.$$

$$a_{2010} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2010x\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos 2010x\,dx$$

$$= -\frac{1}{1005\pi}\int_{0}^{\pi} 2x\sin 2010x\,dx$$

$$= \frac{1}{1005^2}$$

f is even, ⇒ $b_{2010} = 0$

$$\therefore a_{2010} + b_{2010} = \frac{1}{1005^2}$$

Question 3(b)

Let

$$f(x) = \begin{cases} \mathbf{1}, & \mathbf{0} < x < 1 \\ \mathbf{2}, & \mathbf{1} < x < 2 \end{cases}$$

Find the exact expression of the 1st 2 non-zero terms in the sine Fourier half range expansion for $f(x)$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \int_0^1 \sin\frac{n\pi x}{2}\,dx + \int_1^2 2\sin\frac{n\pi x}{2}\,dx$$

$$= \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^1 + \left[-\frac{4}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2$$

$$= \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 2\cos n\pi + 1\right)$$

$$b_1 = \frac{6}{\pi}, \qquad b_2 = -\frac{2}{\pi}$$

$$\therefore f(x) \approx \frac{6}{\pi}\sin\frac{\pi}{2}x - \frac{2}{\pi}\sin \pi x + \cdots$$

Question 4(a)

Let S be the plane which passes through the points $(1,0,0),(2,1,0)$ and $(3,2,1)$. Let L be the line which passes through $(0,0,0)$ and is parallel to the vector $-3\hat{\imath} + \hat{\jmath} - \frac{26}{5}\hat{k}$. Find the coordinates of the point of intersection of L and S.

$$\vec{u} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \vec{v} = \begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

Normal to S,

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 1 & 0\\ 2 & 2 & 1\end{vmatrix} = \hat{\imath} - \hat{\jmath}$$

$$S : x - y = 1, \qquad L : \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3t\\ t\\ -\frac{26}{5}t\end{pmatrix}$$

$$-3t - t = 1, \qquad t = -\frac{1}{4}$$

$$\therefore \textit{point of intersection}, \begin{pmatrix}\frac{3}{4}\\ -\frac{1}{4}\\ \frac{13}{10}\end{pmatrix}$$

Question 4(b)

A space curve C is defined by the vector parametric equation

$$\vec{r}(t) = 2t^2\hat{\imath} + (t^2 - 4t)\hat{\jmath} + (3t - 5)\hat{k}.$$

Let T denote the tangent vector to C at the point corresponding to $t = 1$. Find the length of the projection of T onto the vector $\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

$\vec{r}'(t) = 4t\hat{\imath} + (2t - 4)\hat{\jmath} + 3\hat{k}$

$\vec{T} = \vec{r}(1) = 4\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

$$\therefore \frac{\left|\vec{T} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right|}{\sqrt{1 + 4 + 4}} = \left|\frac{4 - 4 + 6}{3}\right| = 2$$

Question 5(a)

Let $f(x, y, z) = xy + yz + zx + 1505$. Find the exact value of the directional derivative of f at the point $(2, 3, 4)$ in the direction of the vector $\vec{u} = \hat{\imath} - \hat{\jmath} - \sqrt{2}\hat{k}$.

$\nabla f = (y + z, x + z, y + x), \qquad \nabla f(2,3,4) = (7,6,5)$

$$\therefore D_{\vec{u}} f(2,3,4) = \nabla f(2,3,4) \cdot \frac{\vec{u}}{|\vec{u}|} = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} \cdot \frac{1}{\sqrt{1 + 1 + 2}} \begin{pmatrix} 1 \\ -1 \\ -\sqrt{2} \end{pmatrix} = \frac{7 - 6 - 5\sqrt{2}}{2} = \frac{1 - 5\sqrt{2}}{2}$$

Question 5(b)

Find the local maximum points, local minimum points, and saddle points, if any, of the function $f(x, y) = xy + (x + y)(120 - x - y)$.

$f = xy + 120x = x^2 - xy + 120y - y^2$

$f_x = 120 - 2x - y, \qquad f_y = 120 - x - 2y$

$f_x = f_y = 0 \quad \Rightarrow \quad x = y = 40$

There is only one critical point, $(40,40)$.

$f_{xx} = -2, \qquad f_{xy} = -1, \qquad f_{yy} = -2$

$f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$

$\therefore$ $(40,40)$ is a local maximum point.

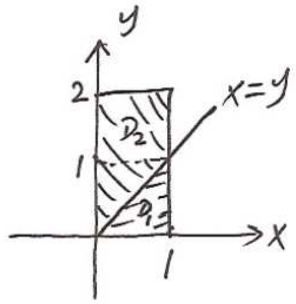
Question 6(a)

Find the exact value of the double integral

$$\iint_D \sqrt{|x-y|}\, dy\, dx$$

where D is the rectangular region $0 \le x \le 1$ and $0 \le y \le 2$.

On D_1, we have $x > y$; on D_2, we have $y > x$.

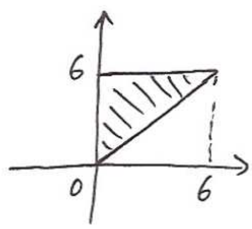


$$\begin{aligned}
\iint_D \sqrt{|x-y|}\, dy\, dx &= \iint_{D_1} \sqrt{x-y}\, dx\, dy + \iint_{D_2} \sqrt{y-x}\, dy\, dx \\
&= \int_0^1 \int_0^x \sqrt{x-y}\, dy\, dx + \int_0^1 \int_x^2 \sqrt{y-x}\, dy\, dx \\
&= \int_0^1 \left[-\frac{2}{3}(x-y)^{\frac{3}{2}}\right]_0^x dx + \int_0^1 \left[\frac{2}{3}(y-x)^{\frac{3}{2}}\right]_x^2 dx \\
&= \int_0^1 \frac{2}{3}x^{\frac{3}{2}}\, dx - \int_0^1 \frac{2}{3}(2-x)^{\frac{3}{2}}\, dx \\
&= \frac{2}{3}\left\{\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^1 - \left[\frac{2}{5}(2-x)^{\frac{5}{2}}\right]_0^1\right\} \\
&= \frac{4}{15} - \frac{4}{15} + \frac{4}{15}\left(2^{\frac{5}{2}}\right) \\
&= \frac{16}{15}\sqrt{2}
\end{aligned}$$

Question 6(b)

Find the exact value of the iterated integral

$$\int_0^6 \int_x^6 \frac{2xy}{\ln\left[(1+y^2)^{1+x^2}\right]}\, dy\, dx.$$



$$\begin{aligned}
\int_0^6 \int_x^6 \frac{2xy}{\ln\left[(1+y^2)^{1+x^2}\right]}\, dy\, dx &= \int_0^6 \int_0^y \frac{2xy}{\ln\left[(1+y^2)^{1+x^2}\right]}\, dx\, dy \\
&= \int_0^6 \frac{y}{\ln(1+y^2)} \left[\ln(1+x^2)\right]_0^y\, dy
\end{aligned}$$

$$= \int_0^6 y\,dy$$

$$= \left[\frac{1}{2}y^2\right]_0^6$$

$$= 18$$

Question 7(a)

Find the exact value of the volume of the solid enclosed laterally by the circular cylinder about z-axis of radius 1, bounded on top by the elliptic paraboloid $2x^2 + 4y^2 + z = 18$ and bounded below by the plane $z = 0$.

Volume,

$$\iint_{x^2+y^2\le 1} 18 - 2x^2 - 4y^2\,dx\,dy = \int_0^{2\pi}\int_0^1 18r - 2r^3\cos^2\theta - 4r^3\sin\theta\,dr\,d\theta$$

$$= 2\pi\int_0^1 18r - 3r^3\,dr$$

$$= 2\pi\left[9r^2 - \frac{3}{4}r^4\right]_0^1$$

$$= \frac{33}{2}\pi$$

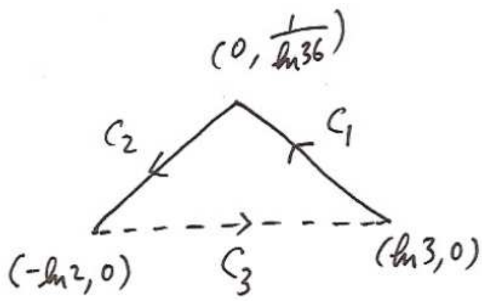
Question 7(b)

Find the exact value of the line integral

$$\int_C (e^x\cos y)dx + (2x - e^x\sin y)dy,$$

where C consists of 2 line segments: C_1 from $(\ln 3, 0)$ to $\left(0, \frac{1}{\ln 36}\right)$, and C_2 from $\left(0, \frac{1}{\ln 36}\right)$ to $(-\ln 2, 0)$.

Apply Green's Theorem to the triangle,



$$\int_{C_1+C_2+C_3} (e^x\cos y)dx + (2x - e^x\sin y)dy$$

$$= \iint_\Delta \frac{\partial}{\partial x}(2x - e^x\sin y) - \frac{\partial}{\partial y}(e^x\cos y)dx\,dy$$

$$= \iint_{\Delta} 2\, dx\, dy$$

$$= 2 \times area\ of\ \Delta$$

$$= 2\left(\frac{1}{2}\right)(\ln 6)\left(\frac{1}{2\ln 6}\right)$$

$$= \frac{1}{2}$$

$$\therefore \int_{C_1+C_2} (e^x \cos y)dx + (2x - e^x \sin y)dy = \frac{1}{2} - \int_{C_3} (e^x \cos y)dx + (2x - e^x \sin y)dy$$

$$= \frac{1}{2} - \int_{\ln 2}^{\ln 3} e^t\, dt$$

$$= \frac{1}{2} - 3 + \frac{1}{2}$$

$$= -2$$

Question 8(a)

Find the exact value of the surface integral

$$\iint_S z\, d\vec{S}$$

where S is the surface $z = x^2 + y^2$ with $0 \le z \le 1$.

$$S : \vec{r}(u,v) = u\hat{\imath} + v\hat{\jmath} + (u^2+v^2)\hat{k}, \qquad 0 \le u^2 + v^2 \le 1$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\hat{\imath} - 2v\hat{\jmath} + \hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^2 + 4v^2 + 1}$$

$$\therefore \iint_S z\, d\vec{S} = \iint_{u^2+v^2\le 1} (u^2+v^2)\sqrt{4u^2+4v^2+1}\, du\, dv$$

$$= \int_0^{2\pi}\int_0^1 r^3\sqrt{4r^2+1}\, dr\, d\theta \qquad \left[let\ t = \sqrt{4r^2+1}\right]$$

$$= 2\pi \int_1^{\sqrt{5}} \frac{t^2-1}{4}\, t\left(\frac{1}{4}t\right) dt$$

$$= \frac{\pi}{8}\int_1^{\sqrt{5}} t^4 - t^2\, dt$$

$$= \frac{\pi}{8}\left[\frac{1}{5}t^5 - \frac{1}{3}t^3\right]_1^{\sqrt{5}}$$

$$= \left(\frac{5\sqrt{5}}{12} + \frac{1}{60}\right)\pi$$

Question 8(b)

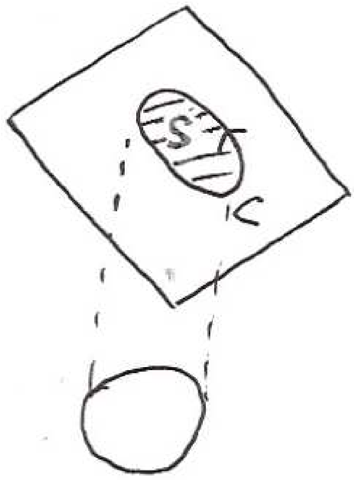
Use Stokes' Theorem to find the exact value of the line integral

$$\oint_C -yz\,dx + xz\,dy + xy\,dz$$

where C is the curve of intersection of the plane $x + y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented in the counterclockwise sense when viewed from above.

Let S = part of the plane $x + y + z = 2$ bounded by C.

$S : \vec{r}(u,v) = u\hat{\imath} + v\hat{\jmath} + (2 - u - v)\hat{k}, \qquad 0 \le u^2 + v^2 \le 1$



$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\nabla \times \left(-yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz & xz & xy \end{vmatrix} = -2y\hat{\imath} + 2z\hat{k}$$

$\vec{r}_u \times \vec{r}_v$ points upwards, so the orientation of S and C are consisitent.

$$\therefore \oint_C -yz\,dx + xz\,dy + xy\,dz = \iint_S \nabla \times \left(-yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}\right) \cdot d\vec{S}$$

$$= \iint_{u^2+v^2 \le 1} -2v + 2(2 - u - v)\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 -4r^2 \sin\theta + 4r - 2r^2 \cos\theta\,dr\,d\theta$$

$$= 2\pi \int_0^1 4r\,dr$$

$$= 2\pi[2r^2]_0^1$$

$$= 4\pi$$