Question 1(a)

Find the slope of the tangent line at the point $(2,-2)$ on the graph of $x^2y^2-2x=4-4y$.

$$x^2y^2-2x=4-4y$$

$$2xy^2+2x^2yy'-2=-4y'$$

$$x=2, y=-2, \qquad \Rightarrow 16-16y'-2=-4y'$$

$$12y'=14, \qquad \Rightarrow y'=\frac{7}{6}$$

Question 1(b)

Find $\frac{1}{\pi}\left(f'(1)-\frac{1}{2\sqrt{3}}\right)$ if $f(x)=x\sin^{-1}\frac{x}{x+1}$.

$$f'(x)=\sin^{-1}\frac{x}{x+1}+\frac{x}{\sqrt{1-\frac{x^2}{(x+1)^2}}}\frac{x+1-x}{(x+1)^2} \quad [\text{for } x>0]$$

$$=\sin^{-1}\frac{x}{x+1}+\frac{x}{(x+1)\sqrt{2x+1}} \quad [\text{for } x>0]$$

$$f'(1)=\sin^{-1}\frac{1}{2}+\frac{1}{2\sqrt{3}}=\frac{\pi}{6}+\frac{1}{2\sqrt{3}}$$

$$\therefore \frac{1}{\pi}f'(1)-\frac{1}{2\sqrt{3}}=\frac{1}{6}$$

Question 2(a)

Given that the function $f(x)=\frac{x(3x-2)}{(x-1)(x-2)}$, where $x\in(1,2)$, attains its absolute maximum value at the point $C\in(1,2)$. Find the value of $(3-\sqrt{2})C$.

$$f(x)=\frac{3x^2-2x}{x^2-3x+2}$$

$$f'(x)=\frac{(x^2-3x+2)(6x-2)-(3x^2-2x)(2x-3)}{(x^2-3x+2)^2}=\frac{-7x^2+12x-4}{(x^2-3x+2)^2}$$

$$f'(x) = 0, \qquad \Rightarrow 7x^2 - 12x + 4 = 0$$

$$x = \frac{12 \pm \sqrt{144 - 112}}{14} = \frac{2}{7}\left(3 \pm \sqrt{2}\right)$$

$$C = \frac{2}{7}\left(3 + \sqrt{2}\right) \quad [C \in (1,2)]$$

$$\therefore \left(3 - \sqrt{2}\right)C = \frac{2}{7}\left(3 + \sqrt{2}\right)\left(3 - \sqrt{2}\right) = 2$$

Question 2(b)

Find the value of

$$\lim_{x \to 0} \frac{\cos^2 8x - \cos^2 5x}{x^2}.$$

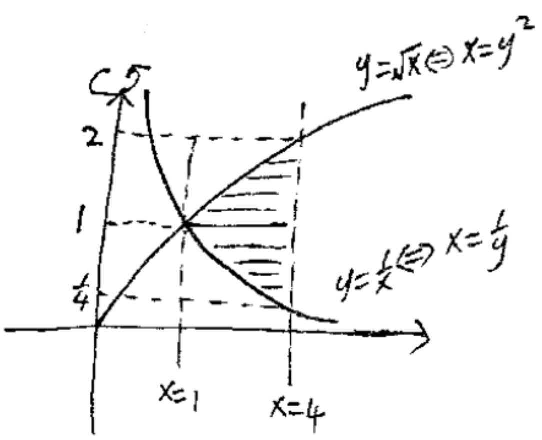
$$\begin{aligned}
\lim_{x \to 0} \frac{\cos^2 8x - \cos^2 5x}{x^2} &= \left(\lim_{x \to 0} \frac{\cos 8x - \cos 5x}{x^2}\right)\left(\lim_{x \to 0}(\cos 8x + \cos 5x)\right) \\
&= 2 \lim_{x \to 0} \frac{-8 \sin 8x + 5 \sin 5x}{2x} \\
&= \lim_{x \to 0}(-64 \cos 8x + 25 \cos 5x) \\
&= -39
\end{aligned}$$

Question 3(a)

Find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{x}, \qquad y = \frac{1}{x}, \qquad x = 1, \qquad x = 4$$

about the y-axis. Give your answer in terms of π.



Volume,

$$\int_{\frac{1}{4}}^{1} \pi\left(16 - \frac{1}{y^2}\right) dy + \int_{1}^{2} \pi(16 - y^4)\, dy$$

$$= \pi\left[16y + \frac{1}{y}\right]_{\frac{1}{4}}^{1} + \pi\left[16y - \frac{1}{5}y^5\right]_{1}^{2}$$

$$= \pi(16 + 1 - 4 - 4) + \pi\left(32 - \frac{32}{5} - 16 + \frac{1}{5}\right)$$

$$= \frac{94}{5}\pi$$

Question 3(b)

Find the value of

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos x \, dx.$$

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos x \, dx = \int_0^{\frac{\pi}{3}} \sin^3 x \, d(\sin x) = \left[\frac{1}{4}\sin^4 x\right]_0^{\frac{\pi}{3}} = \frac{9}{64}$$

Question 5(b)

Let

$$f(x) = \int_0^{x^2} \tan^{-1} t \, dt.$$

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-1)^n$ be the Taylor series representation for $f(x)$ about the point $a = 1$. Find the value of c_2.

$$f'(x) = 2x \tan^{-1} x^2$$

$$f''(x) = 2\tan^{-1} x^2 + \frac{2x \cdot 2x}{1+x^4}$$

$$f''(1) = 2\left(\frac{\pi}{4}\right) + 2 = \frac{\pi}{2} + 2$$

$$\therefore c_2 = \frac{f''(1)}{2!} = \frac{\pi}{4} + 1$$

Question 6(a)

Let

$$f(x) = \begin{cases} 0 & if -2\pi < x < 0 \\ x^2 & if\ 0 < x < 2\pi. \end{cases}$$

Find the coefficient of $\cos x$ in the Fourier Series representation for $f(x)$.

Let $2L = \text{period of f.}$

$$2L = 4\pi, \qquad \Rightarrow L = 2\pi$$

$$\cos\frac{n\pi x}{L} = \cos x, \qquad \Rightarrow \frac{n\pi}{L} = 1, \qquad \Rightarrow n = 2$$

$$
\begin{aligned}
a_2 &= \frac{1}{L}\int_{-L}^{L} f(x)\cos x\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x^2\cos x\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x^2\,d(\sin x) \\
&= \frac{1}{2\pi}\left\{[x^2\sin x]_0^{2\pi} - 2\int_0^{2\pi} x\sin x\,dx\right\} \\
&= \frac{1}{\pi}\int_0^{2\pi} x\,d(\cos x) \\
&= \frac{1}{\pi}\left\{[x\cos x]_0^{2\pi} - \int_0^{2\pi}\cos x\,dx\right\} \\
&= 2
\end{aligned}
$$

Question 6(b)

Let $f(x) = \cos x$ for $0 < x < \pi$. Let $\sum_{n=1}^{\infty} b^n \sin nx$ be the Fourier Sine Series which represents $f(x)$. Find the value of $b_1 + b_2$.

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = \frac{1}{\pi}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi} = 0$$

$$b_2 = \frac{2}{\pi}\int_0^{\pi}\cos x\sin 2x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 3x + \sin x\,dx = \frac{1}{\pi}\left[-\frac{1}{3}\cos 3x - \cos x\right]_0^{\pi} = \frac{8}{3\pi}$$

$$\therefore b_1 + b_2 = \frac{8}{3\pi}$$

Question 7(a)

Solve the differential equation

$$x\frac{dy}{dx} - y = 2x^2\sin 2x$$

with the initial condition $y = \pi$ when $x = \pi$.

$$u\frac{dy}{dx} - u\frac{y}{x} = \frac{d}{dx}(uy) = u\frac{dy}{dx} + \frac{du}{dx}y = 2ux\sin 2x$$

$$\frac{du}{dx} = -\frac{u}{x} \quad\Rightarrow\quad \frac{du}{u} = -\frac{dx}{x} \quad\Rightarrow\quad \ln u = -\ln x \quad\Rightarrow\quad u = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = 2\frac{1}{x}x\sin 2x = 2\sin\ 2x \quad\Rightarrow\quad \frac{1}{x}y = -\cos 2x + c$$

$$y(\pi) = \pi \quad \Rightarrow \quad 1 = -\cos 2\pi + c \quad \Rightarrow \quad c = 2$$

$$\therefore y = x(2 - \cos 2x)$$

Question 7(b)

Solve the differential equation

$$\boldsymbol{y' = \frac{x^2 + xy + y^2}{x^2}}$$

with the initial condition $\boldsymbol{y = 0}$ when $\boldsymbol{x = 1}$.

$$y = vx \quad \Rightarrow \quad y' = v'x + v$$

$$v'x + v = \frac{x^2 + vx^2 + v^2x^2}{x^2} = 1 + v + v^2$$

$$\frac{dv}{1 + v^2} = \frac{dx}{x}$$

$$\tan^{-1} v = \ln|x| + c \quad \Rightarrow \quad \tan^{-1}\frac{y}{x} = \ln|x| + c$$

$$y(1) = 0 \quad \Rightarrow \quad c = 0$$

$$\therefore y = x\tan(\ln|x|)$$

Question 8(a)

Solve the differential equation

$$\boldsymbol{9y'' - 6y' + y = 0}$$

with the initial condition that $\boldsymbol{y = 1}$ and $\boldsymbol{y' = 3}$ when $\boldsymbol{x = 0}$.

$$9\lambda^2 - 6\lambda + 1 = 0 \quad \Rightarrow \quad (3\lambda - 1)^2 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{3}$$

$$y = Ae^{\frac{1}{3}x} + Bxe^{\frac{1}{3}x}$$
$$y' = \frac{1}{3}Ae^{\frac{1}{3}x} + Be^{\frac{1}{3}x} + \frac{1}{3}Bxe^{\frac{1}{3}x}$$

$$y(0) = 1, y'(0) = 3 \quad \Rightarrow \quad A = 1, \qquad B = \frac{8}{3}$$

$$\therefore y = e^{\frac{1}{3}x} + \frac{8}{3}xe^{\frac{1}{3}x}$$

Question 8(b)

Solve the differential equation

$$\boldsymbol{y'' - 5y' + 6y = 18x^2}$$

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = 2,3$$

$$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

$$2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = 18x^2 \quad \Rightarrow \quad A = 3, \qquad B = 5, \qquad C = \frac{19}{6}$$

$$\therefore y = De^{2x} + Ee^{3x} + 3x^2 + 5x + \frac{19}{6}$$

Question 9(a)

Find the Laplace transform $\boldsymbol{L(t\cos 2t)}$.

$$\begin{aligned} f(t) &= t\cos 2t\,, \qquad f(0) = 0 \\ f'(t) &= \cos 2t - 2t\sin 2t\,, \qquad f'(0) = 1 \\ f''(t) &= -4\sin 2t - 4t\cos 2t \end{aligned}$$

$$L(f'') = s^2 L(f) - sf(0) - f'(0)$$

$$-4L(\sin 2t) - 4L(t\cos 2t) = s^2 L(t\cos 2t) - 1$$

$$(s^2 + 4)L(t\cos 2t) = 1 - 4L(\sin 2t) = 1 - \frac{8}{s^2 + 4} = \frac{s^2 - 4}{s^2 + 4}$$

$$L(t\cos 2t) = \frac{s^2 - 4}{(s^2 + 4)^2}$$

Question 9(b)

Find the inverse Laplace transform

$$\boldsymbol{L^{-1}\left[\frac{2s^2 - 4}{(s-2)(s+1)(s-3)}\right].}$$

$$\frac{2s^2 - 4}{(s-2)(s+1)(s-3)} = -\frac{4}{3}\frac{1}{s-2} - \frac{1}{6}\frac{1}{s+1} + \frac{7}{2}\frac{1}{s-3}$$

$$\therefore L^{-1}\left[\frac{2s^2-4}{(s-2)(s+1)(s-3)}\right]=L^{-1}\left[-\frac{4}{3}\frac{1}{s-2}-\frac{1}{6}\frac{1}{s+1}+\frac{7}{2}\frac{1}{s-3}\right]=-\frac{4}{3}e^{2t}-\frac{1}{6}e^{-t}+\frac{7}{2}e^{3t}$$

Question 10(a)

Solve the differential equation

$$\frac{d^2x}{dt^2}=12(t-1)^2U(t-1),\qquad x'(0)=x(0)=0$$

where $U(t-1)=\begin{cases}0 & \text{if } t<1\\ 1 & \text{if } t>1\end{cases}$.

$$L\left(\frac{d^2x}{dt^2}\right)=L[12(t-1)^2U(t-1)]$$

$$s^2\tilde{x}-sx(0)-x'(0)=12e^{-s}\left(\frac{2!}{s^3}\right)$$

$$\tilde{x}=\frac{24}{s^5}e^{-s}$$

$$\therefore x=L^{-1}\left[\frac{24}{s^5}e^{-s}\right]=(t-1)^4U(t-1)$$

Question 10(b)

Find the functions $x(t)$ and $y(t)$ which satisfy

$$\frac{dx}{dt}-y=\frac{t^2}{2},\qquad x-\frac{dy}{dt}=0$$

and the initial conditions $x(0)=0,\ y(0)=1$.

Taking the Laplace transform, we have

$$s\tilde{x}-x(0)-\tilde{y}=\frac{1}{s^3}\quad\Rightarrow\quad s\tilde{x}-\tilde{y}=\frac{1}{s^3}$$
$$\tilde{x}-[s\tilde{y}-y(0)]=0\quad\Rightarrow\quad \tilde{x}-s\tilde{y}=-1$$

$$\tilde{x}=\frac{\begin{vmatrix}\frac{1}{s^3} & -1\\ -1 & -s\end{vmatrix}}{\begin{vmatrix}s & -1\\ 1 & -s\end{vmatrix}}=\frac{-\frac{1}{s^2}-1}{-s^2+1}=\frac{s^2+1}{s^2(s^2-1)}=\frac{1}{s-1}-\frac{1}{s+1}-\frac{1}{s^2}$$

$$x=L^{-1}(\tilde{x})=e^t-e^{-t}-t$$
$$y=\frac{dx}{dt}-\frac{t^2}{2}=e^t+e^{-t}-1-\frac{t^2}{2}$$