Question 1(a)

For what value of m is the line $y = mx + c$ perpendicular to the tangent line of the graph of the function $y = \sqrt{x^2 + 16}$ at the point $(3, 5)$?

$$\frac{dy}{dx}(x) = \frac{x}{\sqrt{x^2 + 16}}$$

$$\frac{dy}{dx}(3) = \frac{3}{5}$$

$$\therefore m = -\frac{1}{3/5} = -\frac{5}{3}$$

Question 1(b)

Find $f'(\sqrt{3})$ if $f(x) = \frac{x(1-x^2)^2}{(\sqrt{1+x^2})}$.

$$\ln f(x) = \ln x + \ln(1 - x^2)^2 - \frac{1}{2}\ln(1 + x^2)$$

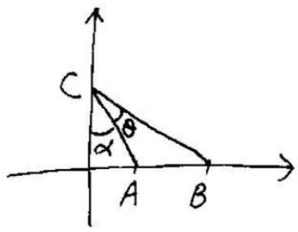
$$\frac{f'(x)}{f(x)} = \frac{1}{x} - \frac{4x}{1 - x^2} - \frac{x}{1 + x^2}$$

$$f'(x) = \frac{(1 - x^2)^2}{\sqrt{1 + x^2}} - \frac{4x^2(1 - x^2)}{\sqrt{1 + x^2}} - \frac{x^2(1 - x^2)^2}{(1 + x^2)^{\frac{3}{2}}}$$

$$f'(\sqrt{3}) = 2 + 12 - \frac{3}{2} = \frac{25}{2}$$

Question 2(a)

2 points A & B start at time $t = 0$ at the origin and move along the positive x-axis with B moving 3 times as fast as A. Let C denote the fixed point $(0, 1)$ on the y-axis. Let θ denote the value of the angle $\angle ACB$ at any time t later. What is the maximum value of $\tan \theta$?



Let A be at the point $(x, 0)$ at time t. So B is at $(3x, 0)$ at time t.

$$\frac{3x}{1} = \tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{\tan\theta + x}{1 - x\tan\theta}$$

$$3x - 3x^2\tan\theta = x + \tan\theta$$

$$\tan\theta = \frac{2x}{3x^2 + 1}$$

$$\frac{d}{dx}(\tan\theta) = \frac{2(3x^2+1) - 2x(6x)}{(3x^2+1)^2} = \frac{2\left(1 - \sqrt{3}x\right)\left(1 + \sqrt{3}x\right)}{(3x^2+1)^2}$$

$$x \geq 0, \qquad x < \frac{1}{\sqrt{3}} \Rightarrow \frac{d}{dx}(\tan\theta) > 0, \qquad x > \frac{1}{\sqrt{3}} \Rightarrow \frac{d}{dx}(\tan\theta) < 0$$

$$\therefore \max \text{of} \tan\theta = \frac{1}{\sqrt{3}}$$

Question 2(b)

Find the value of

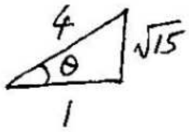
$$\lim_{x \to 3^+} \frac{x^2 \int_3^x \sqrt{t^3 + 9}\, dt}{|3 - x|}$$

$$\begin{aligned} \lim_{x \to 3^+} \frac{x^2 \int_3^x \sqrt{t^3 + 9}\, dt}{|3 - x|} &= \lim_{x \to 3^+} \frac{x^2 \int_3^x \sqrt{t^3 + 9}\, dt}{x - 3} \\ &= \lim_{x \to 3^+} \frac{2x \int_3^x \sqrt{t^3 + 9}\, dt + x^2\sqrt{x^3 + 9}}{1} \\ &= 54 \end{aligned}$$

Question 3(a)

Let θ be the angle in the first quadrant such that $\sin\theta = \frac{\sqrt{15}}{4}$. Find the value of

$$\int_0^\theta \frac{80\sin^3 x}{\sqrt{\cos x}}\,dx$$



$$\begin{aligned}
\int_0^\theta \frac{80\sin^3 x}{\sqrt{\cos x}}\,dx &= -\int_0^\theta \frac{80\sin^2 x}{\sqrt{\cos x}}\,d(\cos x)\\
&= -80\int_0^\theta \frac{1}{\sqrt{\cos x}} - (\cos x)^{\frac{3}{2}}\,d(\cos x)\\
&= -80\left[2\sqrt{\cos x} - \frac{2}{5}(\cos x)^{\frac{5}{2}}\right]_0^\theta\\
&= 49
\end{aligned}$$

Question 3(b)

Let $x > 1$. Find

$$\int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right) dx.$$

$$\begin{aligned}
\int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right) dx &= \int \frac{1}{\ln x}\,dx - \int \frac{1}{(\ln x)^2}\,dx\\
&= \frac{x}{\ln x} + \int x\frac{1}{(\ln x)^2}\frac{1}{x}\,dx - \int \frac{1}{(\ln x)^2}\,dx\\
&= \frac{x}{\ln x} + c
\end{aligned}$$

Question 4(a)

By using the Ratio Test, or otherwise, determine whether the series

$$\sum_{n=1}^{\infty} \frac{6^n (n!)^2}{(2n)!}$$

is convergent or divergent. Show clearly all your steps.

$$\lim_{n\to\infty} \frac{6^{n+1}[(n+1)!]^2}{(2n+2)!} \frac{(2n)!}{6^n(n!)^2} = \lim_{n\to\infty} \frac{6(n+1)^2}{(2n+2)(2n+1)} = \frac{3}{2} > 1$$

$$\therefore \text{the series is divergent.}$$

Question 5(b)

Let

$$\boldsymbol{f(x) = \frac{1}{x^2 + x + 1}.}$$

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be the Maclaurin series representation for $f(x)$. Find the value of $c_{36} - c_{37} + c_{38}$.

$$f(x) = \frac{1}{x^2+x+1} = \frac{1-x}{1-x^3} = (1-x)\sum_{n=0}^{\infty} x^{3n} = \sum_{n=0}^{\infty} x^{3n} - \sum_{n=0}^{\infty} x^{3n+1}$$

$$c_{36} = c_{3\cdot 12} = 1, \qquad c_{37} = c_{3\cdot 12+1} = -1, \qquad c_{38} = c_{3\cdot 12+2} = 0$$

$$\therefore c_{36} - c_{37} + c_{38} = 2$$

Note: This solution will only be valid for $-1 < x < 1$.

Question 6(a)

Let

$$f(x) = \begin{cases} \boldsymbol{0} & \boldsymbol{if -1 < x < 0} \\ \boldsymbol{1} & \boldsymbol{if\ 0 < x < 1.} \end{cases}$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$ be the Fourier Series representation for $f(x)$. Find the value of $a_0 - \pi a_3 + \pi b_5$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{0}^{1} 1\,dx = \frac{1}{2}$$

$$a_3 = \frac{1}{1}\int_{-1}^{1} f(x)\cos 3\pi x\,dx = \int_{0}^{1} \cos 3\pi x\,dx = \left[\frac{\sin 3\pi x}{3\pi}\right]_0^1 = 0$$

$$b_5 = \frac{1}{1}\int_{-1}^{1} f(x)\sin 5\pi x\,dx = \int_{0}^{1} \sin 5\pi x\,dx = \left[-\frac{\cos 5\pi x}{5\pi}\right]_0^1 = \frac{2}{5\pi}$$

$$\therefore a_0 - \pi a_3 + \pi b_5 = \frac{1}{2} + \frac{2}{5} = \frac{9}{10}$$

Question 6(b)

Let $f(x) = x(\pi - x)$ for $0 < x < \pi$. Let $\sum_{n=1}^{\infty} b_n \sin n\pi x$ be the Fourier Sine Series which represents $f(x)$. Find the value of the coefficient b_3. Give your answer in terms of π.

$$\begin{aligned}
b_3 &= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin 3x\,dx \\
&= 2\int_0^{\pi} x\sin 3x\,dx - \frac{2}{\pi}\int_0^{\pi} x^2 \sin 3x\,dx \\
&= 2\int_0^{\pi} -\frac{1}{2}x\,d(\cos 3x) + \frac{2}{\pi}\int_0^{\pi}\frac{1}{3}x^2\,d(\cos 3x) \\
&= \left[-\frac{2}{3}x\cos 3x\right]_0^{\pi} + \frac{2}{3}\int_0^{\pi}\cos 3x\,dx + \left[\frac{2}{3\pi}x^2\cos 3x\right]_0^{\pi} - \frac{2}{3\pi}\int_0^{\pi} 2x\cos 3x\,dx \\
&= \frac{2\pi}{3} + \left[\frac{2}{9}\sin 3x\right]_0^{\pi} - \frac{2\pi}{3} - \frac{2}{3\pi}\int_0^{\pi}\frac{2}{3}x\,d(\sin 3x) \\
&= \left[-\frac{4}{9\pi}x\sin 3x\right]_0^{\pi} + \frac{4}{9\pi}\int_0^{\pi}\sin 3x\,dx \\
&= \left[-\frac{4}{27\pi}\cos 3x\right]_0^{\pi} \\
&= \frac{8}{27\pi}
\end{aligned}$$

Question 7(a)

Find $f(x)$ which satisfies the differential equation

$$f'(x) + \frac{2}{x}f(x) = 8x, \qquad x > 0$$

and the initial condition $f(1) = 3$.

$$uf' + u\left(\frac{2}{x}\right)f = \frac{d}{dx}(uf) = u'f + uf' = 8ux$$

$$u' = \frac{2u}{x} \quad \Rightarrow \quad \frac{du}{u} = \frac{2\,dx}{x}$$

$$\ln u = 2\ln x = \ln x^2 \quad \Rightarrow \quad u = x^2$$

$$\frac{d}{dx}(x^2 f) = 8x^3 \quad \Rightarrow \quad x^2 f = 2x^4 + c$$

$$f(1) = 3 \quad \Rightarrow \quad 3 = 2 + c \quad \Rightarrow \quad c = 1$$

$$\therefore f(x) = \frac{2x^4+1}{x^2} = 2x^2 + \frac{1}{x^2}$$

Question 7(b)

Solve the differential equation

$$\boldsymbol{y' = \frac{2y^4+x^4}{xy^3}, \qquad y(1) = 2.}$$

$$Let\ y = ux$$

$$u'x + u = \frac{2u^4x^4+x^4}{u^3x^4} = \frac{2u^4+1}{u^3} \quad \Rightarrow \quad u'x = \frac{u^4+1}{u^3}$$

$$\frac{u^3}{u^4+1}du = \frac{dx}{x}$$

$$\frac{1}{4}\ln(u^4+1) = \ln x + c_1 \quad \Rightarrow \quad u^4+1 = cx^4$$

$$y^4 + x^4 = cx^8$$

$$y(1) = 2 \quad \Rightarrow \quad 16+1 = c \quad \Rightarrow \quad c = 17$$

$$\therefore y^4 = 17x^8 - x^4$$

Question 8(a)

Find the general solution of the differential equation

$$\boldsymbol{y'' - 2y' + 10y = 0.}$$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$$

$$\therefore y = e^x(c_1\cos 3x + c_2 \sin 3x)$$

Question 8(b)

Solve the differential equation

$$\boldsymbol{y'' - y = e^x, \qquad y(0) = 2, \qquad y'(0) = \frac{1}{2}.}$$

$$\lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda = \pm 1$$

General solution,

$$y = c_1 e^x + c_2 e^{-x}$$

Particular integral,

$$y = Axe^x$$
$$y' = Axe^x + Ae^x$$
$$y'' = Axe^x + 2Ae^x$$

$$y'' - y = e^x \quad \Rightarrow \quad 2Ae^x = e^x \quad \Rightarrow \quad A = \frac{1}{2}$$

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}xe^x$$
$$y' = c_1 e^x - c_2 e^x + \frac{1}{2}xe^x + \frac{1}{2}e^x$$

Using $y(0) = 2,\ y'(0) = \frac{1}{2}$,

$$c_1 + c_2 = 2$$
$$c_1 + c_2 + \frac{1}{2} = \frac{1}{2}$$

$$c_1 = c_2 = 1$$

$$\therefore y = e^x + e^{-x} + \frac{1}{2}xe^x$$

Question 9(a)

A tank initially holds 100l of a brine solution containing 20kg of salt. Starting from time $t = 0$, fresh water is continuously poured into the tank at the rate of $5l/\text{min}$, while the well-stirred mixture leaves the tank continuously at the same rate. Find an expression for the amount (in kg) of salt in the tank at any time t.

Let Qkg be the amount of salt at time t.

$$\frac{dQ}{dt} = -\frac{5}{100}Q = -\frac{Q}{20}$$

$$\frac{dQ}{Q} = -\frac{1}{20}dt \quad \Rightarrow \quad \ln Q = -\frac{1}{20}t + c \quad \Rightarrow \quad Q = ce^{-\frac{t}{20}}$$

At $t = 0$, we have $Q = 20$, $\quad \Rightarrow \quad c = 20$

$$\therefore Q = 20e^{-\frac{t}{20}}$$

Question 9(b)

Find the inverse Laplace transform

$$\boldsymbol{L^{-1}\left[\frac{1}{s(s^2+4)}\right].}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$1 = A(s^2+4) + Bs^2 + Cs = (A+B)s^2 + Cs + 4A \quad \Rightarrow \quad A = \frac{1}{4}, \qquad B = -\frac{1}{4}, \qquad C = 0$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+4)}\right] = L^{-1}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right] = \frac{1}{4} - \frac{1}{4}\cos 2t$$

Question 10(a)

Given that the Laplace transform of $\sqrt{t}$ is $L(\sqrt{t}) = \frac{1}{2}\sqrt{\pi}s^{-\frac{3}{2}}$. Find the Laplace transform of the function $f(t) = t^{\frac{3}{2}}$.

$$\int_0^t \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^t = \frac{2}{3}t^{\frac{3}{2}}$$

$$L\left[\int_0^t \sqrt{u}\,du\right] = \frac{L(\sqrt{t})}{s} = \frac{1}{2}\sqrt{\pi}s^{-\frac{5}{2}}$$

$$L\left(\frac{2}{3}t^{\frac{3}{2}}\right) = \frac{1}{2}\sqrt{\pi}s^{-\frac{5}{2}} \quad \Rightarrow \quad L\left(t^{\frac{3}{2}}\right) = \frac{3}{4}\sqrt{\pi}s^{-\frac{5}{2}}$$

Question 10(b)

Find the functions $x(t)$ and $y(t)$ which satisfy

$$\frac{dx}{dt} = -2x + y, \qquad \frac{dy}{dt} = 2x - 3y$$

and the initial conditions $x(0) = 1,\ y(0) = 0$.

Taking Laplace transform with $L(x) = X,\ L(y) = Y$, we have

$$\begin{cases} L\left(\dfrac{dx}{dt}\right) = -2X + Y \\ L\left(\dfrac{dy}{dt}\right) = 2X - 3Y \end{cases} \Rightarrow \quad \begin{cases} sX - x(0) = -2X + Y \\ sY - y(0) = 2X - 3Y \end{cases}$$

$$\begin{cases} (s+2)X - Y = 1 \\ -2X + (s+3)Y = 0 \end{cases}$$

$$\begin{cases} X = \dfrac{s+3}{s^2+5s+4} = \dfrac{2}{3}\dfrac{1}{s+1} + \dfrac{1}{3}\dfrac{1}{s+4} \\ Y = \dfrac{2}{s^2+5s+4} = \dfrac{2}{3}\dfrac{1}{s+1} - \dfrac{2}{3}\dfrac{1}{s+4} \end{cases}$$

$$\begin{cases} x = \dfrac{2}{3}e^{-t} + \dfrac{1}{3}e^{-4t} \\ y = \dfrac{2}{3}e^{-t} - \dfrac{2}{3}e^{-4t} \end{cases}$$

Solutions provided by : **NUS Mathematics Department**
Solutions typed by :**John Soo Yue Han**

Updates:
18/8/2012 **Lee Yuan Zhe**: Q2a, Q5b, Q9b