PC1221/2
Fundamentals of Physics I/II

Semester I 2007/08

Measurements and Error Analysis

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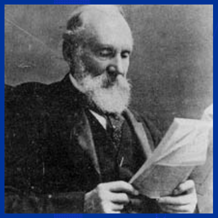
Outline

- Rules of the Game
- Errors in measurements
- Accuracy and Precision
- Systematic and Random Errors
- Statistical languages
  - ✓ mean
  - ✓ standard deviation
  - ✓ standard error
- Error Propagation
- Linear Least Squares Fit

What is measurement?

Measurement is the process of quantifying experience of the external world.

"when you can measure what you are speaking about and express it in numbers, you know something about it; but, when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts to the stage of science."



Lord Kelvin

Pioneer of thermodynamics & statistical mechanics

Uncertainties and/or Errors

- All measurements have some degree of uncertainties/errors.
- Error is the difference between the result of the measurement and the true value.
- The study and evaluation of error in measurement is often called error analysis.
- The complete statement of a measured value SHOULD include an estimate of its error.

Expressing uncertainties

The result of a measurement is presented as

(best estimate ± uncertainty) units

$$(x_{\text{best}} \pm dx)\,\text{units}$$

We are quite confident that the quantity lies within

$$x_{\text{best}} - dx < x < x_{\text{best}} + dx$$

Example: $g \pm \Delta g = (9.803 \pm 0.008)\ \text{m/s}^2$

Absolute and Fractional uncertainty

(best estimate ± uncertainty) units

$$(x_{\text{best}} \pm dx)\,\text{units}$$

Absolute uncertainty: dx

It represents the actual amount by which the best estimated value is uncertain.

Fractional uncertainty: $\dfrac{dx}{x_{\text{best}}}$

It gives us the significance of the uncertainty with respect to the best estimated value.

Accuracy and Precision

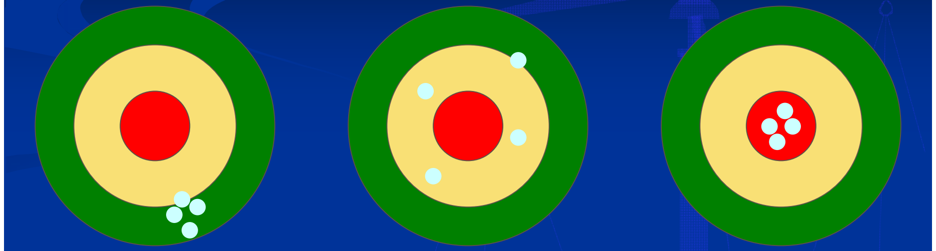
Accuracy

- a measure of how close an experimental result is to the “true” (or published or accepted) value.

Precision

- a measure of the degree of closeness of repeated measurements.

Accuracy versus Precision



POOR accuracy	POOR accuracy	GOOD accuracy
GOOD precision	POOR precision	GOOD precision

Accuracy and Precision

Consider the two measurements:

A = (2.52 ± 0.02) cm

B = (2.58 ± 0.05) cm

Which is more precise?

Which is more accurate?

Types of Uncertainties

Random Errors

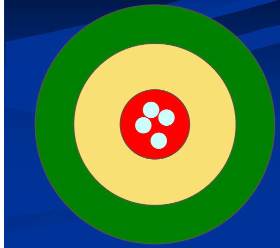
- ✓ Results from unknown and unpredictable variations that arise in all experimental situations.
- ✓ Repeated measurements will give slightly different values each time.
- ✓ You cannot determine the magnitude (size) or sign of random uncertainty from a single measurement.
- ✓ Random errors can be estimated by taking several measurements.

Types of Uncertainties

Systematic Errors

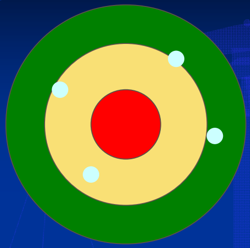
- ✓ Associated with particular measurement instruments or techniques.
- ✓ The same sign and nearly the same magnitude of the error is obtained on repeated measurements.
- ✓ Commonly caused by improperly “calibrated” or “zeroed” instrument or by experimenter bias.
- ✓ Systematical errors cannot be eliminated by averaging or treated statistically.

Systematic versus Random Errors



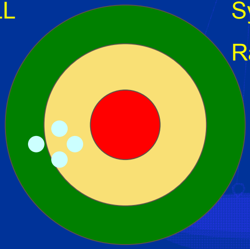
Systematic: SMALL

Random: SMALL



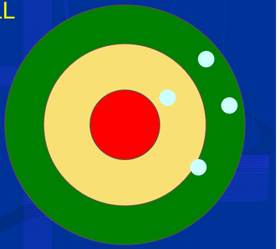
Systematic: SMALL

Random: LARGE



Systematic: LARGE

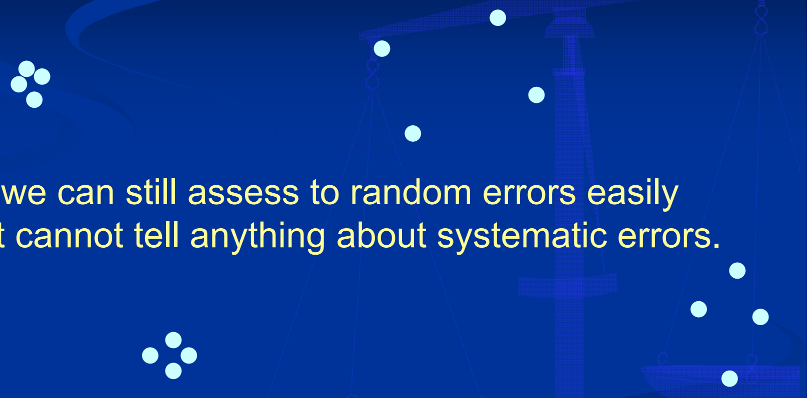
Random: SMALL



Systematic: LARGE

Random: LARGE

True value is generally not known…



… we can still assess to random errors easily
but cannot tell anything about systematic errors.

Significant figures

In a measured quantity, all digits are significant except any zeros whose sole purpose is to show the location of the decimal place.

123	g	________	1.23×10^{2} g
123.0	g	________	1.230×10^{2} g
0.0012	m	________	1.2×10^{-3} m
0.0001203	cm	________	1.203×10^{-4} s
0.001230	s	________	1.230×10^{-4} s
1000	cm	________	1×10^{3} cm
1000.	cm	________	1.000×10^{3} cm
150		________	150

Expressing uncertainty

When expressing a measurement and its associated uncertainty as

(measured value ± uncertainty) units

- ❖Round the uncertainty to one significant figure, then
- ❖round the measurement to the same precision as the uncertainty.

For example, round 9.802562 ± 0.007916 m/s^2 to

$g \pm \Delta g = (9.803 \pm 0.008)$ m/s^2

Significant figures in calculations

Addition and Subtraction

When adding or subtracting physical quantities, the precision of the final result is the same as the precision of the least precise term.

$$\begin{array}{rl} 132.45 & \text{cm} \\ 0.823 & \text{cm} \\ \underline{+\ 5.6} & \underline{\text{cm}} \\ 138.873 & \text{cm} \quad \text{-->}\ 138.9\ \text{cm} \end{array}$$

Significant figures in calculations

Multiplication and Division

When multiplying or dividing physical quantities, the number of significant digits in the final result is the same as the factor (or divisor…) with the fewest number of significant digits.

6.273 N	0.0204 mm
× 5.5 m	÷ 21 C°
34.5015 N·m	0.00097142857 mm/C°
35 N·m	0.00097 mm/C°

Mean (Average) value

Let $x_1, x_2, \ldots x_N$ represent a set of N measurements of a physical quantity x.

The average or mean value of this set of measurements is given by

$$\boxed{\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{N}\left(x_1 + x_2 + \ldots + x_N\right)}$$

The mean is *almost* entirely free from random errors and gives the best estimate for the value of the quantity measured for a large number of readings.

Standard deviation

Standard deviation quantifies the spread of the data about the mean.

$$\boxed{s_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2}}$$

Statistical Interpretation:

- ✓ 68.3% within 1σ
- ✓ 95.5% within 2σ
- ✓ 99.73% within 3σ

Standard deviation as uncertainty

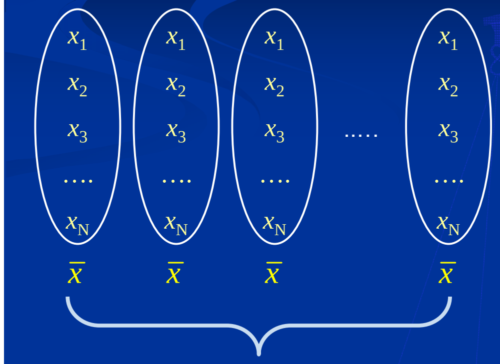
Suppose the mean and standard deviation of a set of N measurements of a physical quantity x have been determined.

If one more **extra** measurement is to be made (under the same conditions), then the reading x_{N+1} would have a probability of 68.3% lying within

$$\boxed{\bar{x} - s_x < x_{N+1} < \bar{x} + s_x}$$

The standard deviation is then treated as the uncertainty for the measurement of a single reading.

Standard deviation of the mean



The overall mean for all means can then be determined.

Standard deviation of the mean measures the spread of all means about the overall mean.

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}}$$

It is also called the standard error and is the error usually quoted for a measurement in the literature.

Expressing uncertainty

(best estimate ± uncertainty) units

$$(x_{\text{best}} \pm dx)\,\text{units}$$

Best estimate: $x_{\text{best}} = \bar{x}$

Uncertainty: $dx = s_{\bar{x}}$

Statistically, the true value would have a probability of 68.3% lying within

$$\bar{x} - s_{\bar{x}} < x_{\text{true}} < \bar{x} + s_{\bar{x}}$$

Standard error as uncertainty

N	Mean	S.D.	S.E.	Result
10	19.6	2.71	0.857	19.6 ± 0.9
100	19.89	2.26	0.226	19.9 ± 0.2
1000	19.884	2.48	0.0784	19.88 ± 0.08
10000	19.9879	2.52	0.0252	19.99 ± 0.03

Example 1

A student measures the resistance of a coil eight times and obtains the results as follow:

$R \pm 0.001$ (Ω)							
4.615	4.638	4.597	4.634	4.613	4.623	4.659	4.623

State the final result of the resistance with the appropriate number of significant figures.

Example 1: Solution

Mean:

$$\bar{R} = \frac{1}{8}\sum_{i=1}^{8} R_i = 4.625\,\Omega$$

Standard deviation:

$$s_R = \sqrt{\frac{1}{8-1}\sum_{i=1}^{8}\left(R_i - \bar{R}\right)^2} = 0.01868\,\Omega$$

Standard error:

$$s_{\bar{R}} = \frac{s_R}{\sqrt{8}} = 0.006603\,\Omega$$

Final result:

$$\bar{R} \pm s_{\bar{R}} = (4.625 \pm 0.007)\,\Omega$$

Error propagation

Suppose the value of a quantity *R(x,y,z,…)* is determined from the measured values of a number of independent quantities *x, y, z, …* which are directly measured.

Question: How to determine the error in *R* from the errors associated with the measurements of *x, y, z, …* respectively?

$$\boxed{\sigma_R = \sqrt{\left(\frac{\partial R}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial R}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial R}{\partial z}\right)^2 \sigma_z^2 + \dots}}$$

Combining uncertainties

Addition and Subtraction

Let *A* ± Δ*A* and *B* ± Δ*B* represent two measured quantities.

The uncertainty in the sum *S* = *A* + *B* is

$$\Delta S = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

The uncertainty in the difference *D* = *A* - *B* is ALSO

$$\Delta D = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

Combining uncertainties

Product and Quotient

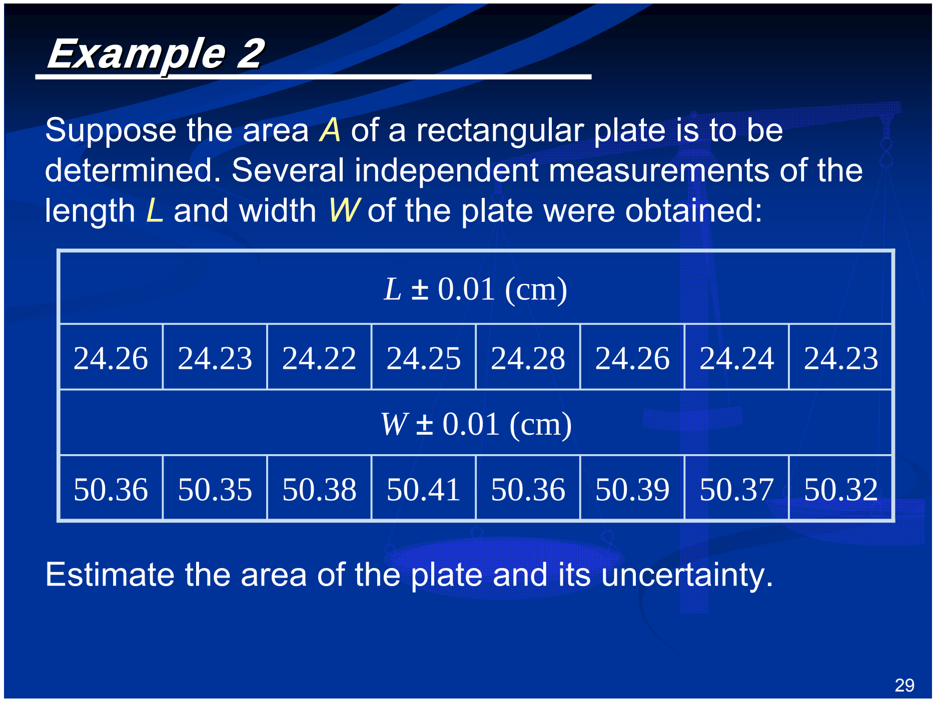
Let *A* ± Δ*A* and *B* ± Δ*B* represent two measured quantities.

The uncertainty in the product *P* = *A* × *B* is

$$\Delta P = P\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$$

The uncertainty in the quotient *Q* = *A* / *B* is ALSO

$$\Delta Q = Q\sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}$$

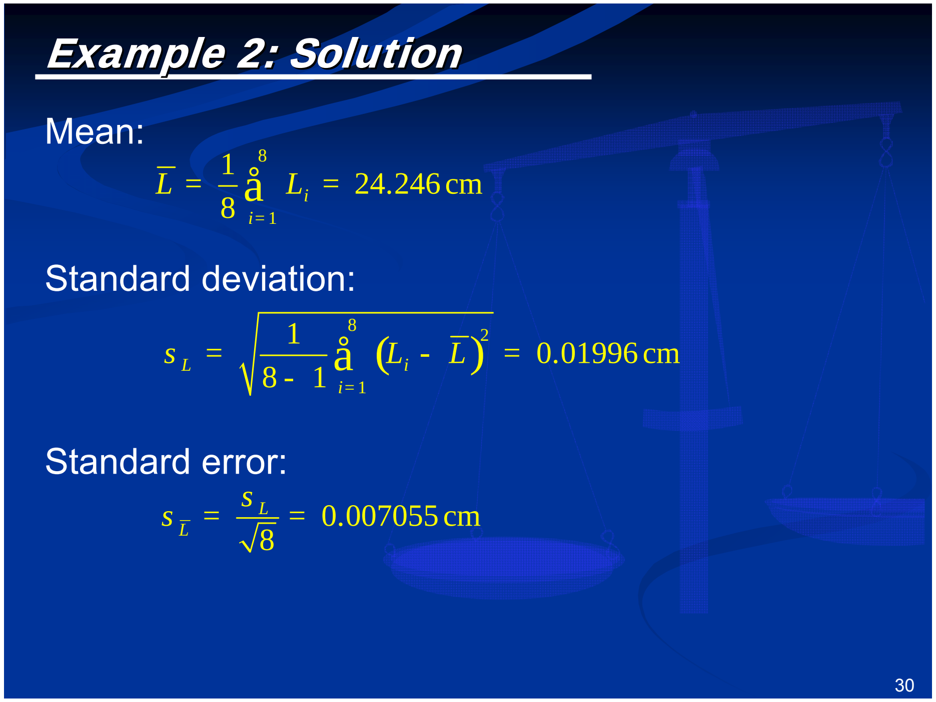


Example 2

Suppose the area *A* of a rectangular plate is to be determined. Several independent measurements of the length *L* and width *W* of the plate were obtained:

$L \pm 0.01$ (cm)							
24.26	24.23	24.22	24.25	24.28	24.26	24.24	24.23
$W \pm 0.01$ (cm)							
50.36	50.35	50.38	50.41	50.36	50.39	50.37	50.32

Estimate the area of the plate and its uncertainty.



Example 2: Solution

Mean:

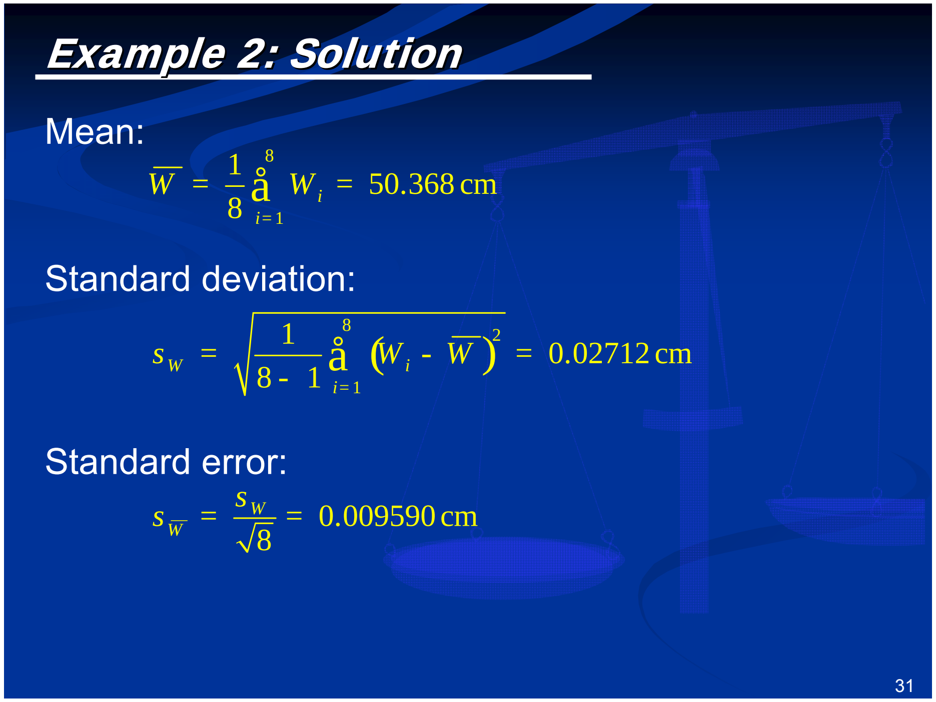
$$\bar{L} = \frac{1}{8}\sum_{i=1}^{8} L_i = 24.246\,\text{cm}$$

Standard deviation:

$$s_L = \sqrt{\frac{1}{8-1}\sum_{i=1}^{8}\left(L_i - \bar{L}\right)^2} = 0.01996\,\text{cm}$$

Standard error:

$$s_{\bar{L}} = \frac{s_L}{\sqrt{8}} = 0.007055\,\text{cm}$$



Example 2: Solution

Mean:

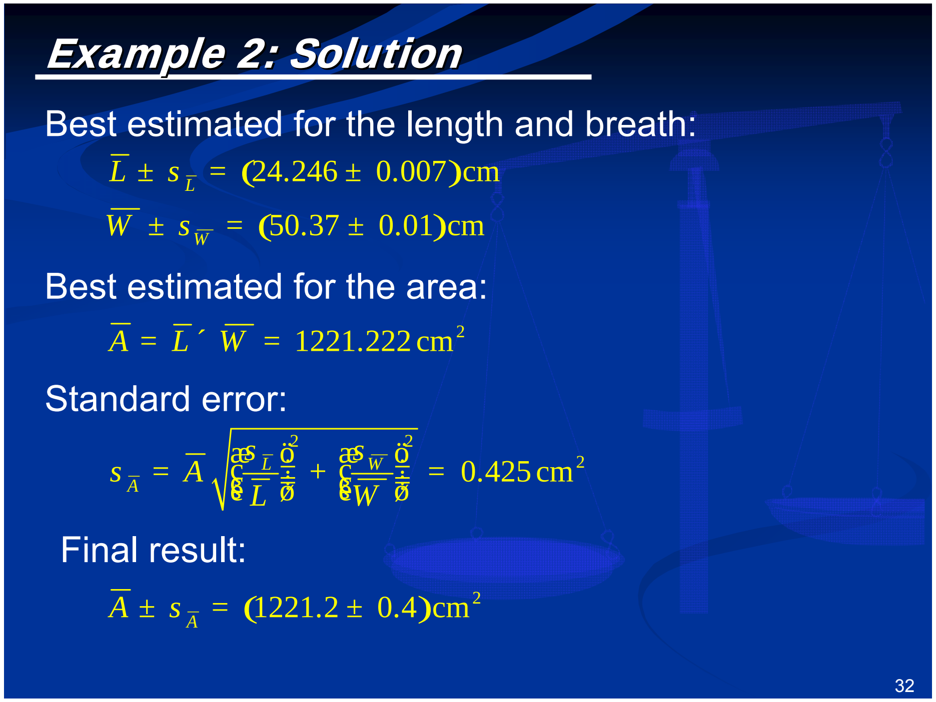
$$\bar{W} = \frac{1}{8}\sum_{i=1}^{8} W_i = 50.368\,\text{cm}$$

Standard deviation:

$$s_W = \sqrt{\frac{1}{8-1}\sum_{i=1}^{8}\left(W_i - \bar{W}\right)^2} = 0.02712\,\text{cm}$$

Standard error:

$$s_{\bar{W}} = \frac{s_W}{\sqrt{8}} = 0.009590\,\text{cm}$$



Example 2: Solution

Best estimated for the length and breath:

$$\bar{L} \pm s_{\bar{L}} = (24.246 \pm 0.007)\,\text{cm}$$

$$\bar{W} \pm s_{\bar{W}} = (50.37 \pm 0.01)\,\text{cm}$$

Best estimated for the area:

$$\bar{A} = \bar{L} \times \bar{W} = 1221.222\,\text{cm}^2$$

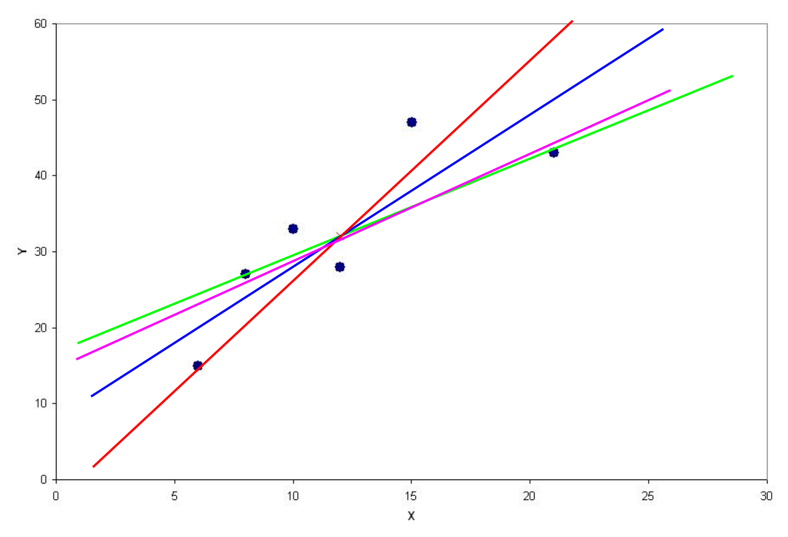
Standard error:

$$s_{\bar{A}} = \bar{A}\sqrt{\left(\frac{s_{\bar{L}}}{\bar{L}}\right)^2 + \left(\frac{s_{\bar{W}}}{\bar{W}}\right)^2} = 0.425\,\text{cm}^2$$

Final result:

$$\bar{A} \pm s_{\bar{A}} = (1221.2 \pm 0.4)\,\text{cm}^2$$

Which is the BEST line?



Least squares fit

The best straight line to fit a set of measured data points (x_1, y_1), (x_2, y_2), …, (x_n, y_n) is

$$y_{\text{best}} = m_{\text{best}}x + c_{\text{best}}$$

Assumption:

- The uncertainty in our measurements of x is negligible but not in y
- The uncertainty in our measurements of y is the same

Least squares fit: Formulas

$$\Delta = n\sum x_i^2 - \left(\sum x_i\right)^2$$

Slope:

$$m_{\text{best}} = \frac{1}{\Delta}\left(n\sum x_i y_i - \sum x_i \sum y_i\right) \qquad \sigma_{m_{\text{best}}} = \sqrt{n\frac{\sigma^2}{\Delta}}$$

y-intercept:

$$c_{\text{best}} = \frac{1}{\Delta}\left(\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i\right) \qquad \sigma_{c_{\text{best}}} = \sqrt{\frac{\sigma^2}{\Delta}\sum x_i^2}$$

Standard deviation for y:

$$\sigma_y = \sqrt{\frac{1}{n-2}\sum\left(y_i - c_{best} - m_{best}x_i\right)^2}$$

Example 3

A student measures the pressure *P* of a gas at five different temperatures *T* by keeping the volume constant. His data are should in the following table:

P ± 5 (mm of mercury)				
65	75	85	95	105
T ± 1 (°C)				
50.36	50.35	50.38	50.41	50.36

$$T = A + BP$$

Estimate the value of absolute zero and its error with appropriate number of significant figures.

Example 3: Solution

slope, m_{best} = 3.71 °C/(mm of mercury)
y-intercept, c_{best} = −263.35 °C
σ_y = 6.6808 °C
σ_m = 0.2113 °C/(mm of mercury)
σ_c = 18.2045 °C

The best fit of a straight line,
T = − 263.35 + 3.71 P

Final result:

$$\bar{A} \pm \sigma_{\bar{A}} = (-263 \pm 8)°\mathrm{C}$$

Example 4

A student wants to measure the acceleration of gravity *g* by measuring the period *T* and the length *ℓ* of a simple pendulum.

ℓ ± 0.1 (cm)				
57.3	61.1	73.2	83.7	95.0
T ± 0.001 (s)				
1.521	1.567	1.718	1.835	1.952

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Estimate the acceleration of gravity *g* and its uncertainty with appropriate number of significant figures.

Example 4: Solution

slope, m_{best} = 0.03987 s^2/cm
y-intercept, c_{best} = 0.0268 s^2
σ_m = 0.0002067 s^2/cm
σ_c = 0.01558 s^2

The best fit of a straight line,
T^2 = 0.03987 ℓ + 0.0268

Example 4: Solution

Acceleration of gravity:

$$g = \frac{4\pi^2}{m} = 990.1748\ \mathrm{cm/s^2}$$

Standard deviation of *g*:

$$\sigma_g = g\frac{\sigma_m}{m} = 5.13274\ \mathrm{cm/s^2}$$

Standard error of *g*:

$$\sigma_{\bar{g}} = \frac{\sigma_g}{\sqrt{n}} = 2.2954\ \mathrm{cm/s^2}$$

Final result:

$$\bar{g} \pm \sigma_{\bar{g}} = (990 \pm 2)\mathrm{cm/s^2}$$