PC2230, Midterm test, 2 Mar 2022, 10:00-11:50 AM, closed book, each sub-question is 7.7 marks

Problem 1. In the course of pumping up a bicycle tire, a liter (10^{-3} m^3) of air at 1 atmospheric pressure is compressed adiabatically to a pressure of 7 atm. Air is mostly diatomic nitrogen (80%) and oxygen (20%). The adiabatic exponent $\gamma = 1 + 2/f$, for f-degree molecule.

(a) What is the final volume of this air after compression?
(b) How much work is done in compressing the air?
(c) If the temperature of the air is initially 300 K, what is the temperature after compression?

Problem 2. This problem gives an approach to estimating the width of the peak of the total multiplicity function for a system of two large Einstein solids.

(a) Consider two identical Einstein solids, each with N oscillators, in thermal contact with each other. Suppose that the total number of energy units in the combined system is exactly $q = 2N$. How many different macrostates (that is, possible values for the total energy the first solid) are there for this combined system?
(b) Using the approximation result for the multiplicity after a better Stirling's approximation, i.e., $\Omega(N, q) \approx \frac{\left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N}{\sqrt{2\pi q(q+N)/N}}$, find an approximate expression for the total number of microstates for the combined system. (Hint: Treat the combined system as a single Einstein solid. Do not throw away factors of "large" numbers, since you will eventually be dividing two "very large" number that are nearly equal).
(c) The most likely macrostate for this system is (of course) the one in which the energy is shared equally between the two solids. Using the expression in (b) to find an approximate expression for the multiplicity of this macrostate.
(d) You can get a rough idea of the "sharpness" of the multiplicity function by comparing your answers to part (b) and (c). Part (c) tells you the height of the peak, while part (b) tells you the total area under the entire graph. As a very crude approximation, pretend that the peak's shape is rectangular. In this case, how wide would it be? Out of all the macrostates, what fraction have reasonable large probabilities? Evaluate this fraction numerically for the case $N = 10^{23}$.

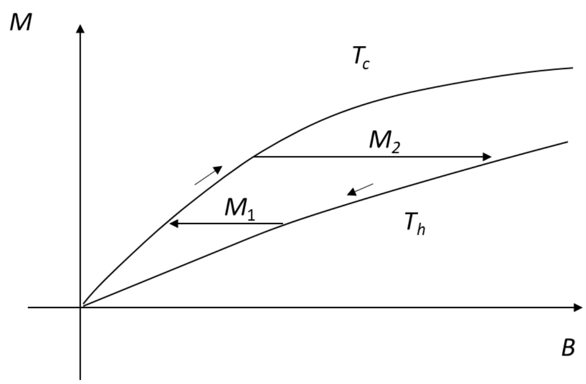
Problem 3. Consider a paramagnet with N spins in total and n for spins up, and $N - n$ for spin down. For each spin with two orientations, the up spin contributes energy of $-\mu B$, and down spin contributes energy $+\mu B$. B is magnetic field and μ is magnetic moment of the spin.

(a) Give the formula for the multiplicity $\Omega(N, n)$, which is the number of microstates with n spins in the up states.
(b) Determine the entropy S as a function of n after using Stirling's approximation.
(c) Give a relation between the total energy U and magnetization M. The magnetization is $M = \mu(\text{number of up spins} - \text{down spins}) = \mu(2n - N)$. Show that the magnetization is given by $M = N\mu \tanh(\frac{\mu B}{kT})$, with temperature T determined from the entropy.

(d) Show that if the magnetization M is a constant, in a M v.s. B diagram, it represents a reversible adiabatic process.

(e) An isothermal process with a fixed temperature T in the diagram is represented by the curve $M = N\mu \tanh(\frac{\mu B}{kT})$. If the magnetization is increased from M_1 to M_2 at constant temperature, will the paramagnet absorb or release heat? What is the amount Q?

(f) We run a loop in the M-B space by first decreasing M from M_2 to M_1 isothermally at a high temperature T_h, and then run horizontally adiabatically, and then, go back to high magnetization M_2 isothermally at T_c, closing the loop by another adiabatic process. What is the efficiency $e = 1 - Q_c/Q_h$ of this Carnot cycle?



Physical Constants

$$k = 1.381 \times 10^{-23}\ \mathrm{J/K}$$
$$= 8.617 \times 10^{-5}\ \mathrm{eV/K}$$
$$N_A = 6.022 \times 10^{23}$$
$$R = 8.315\ \mathrm{J/mol{\cdot}K}$$
$$h = 6.626 \times 10^{-34}\ \mathrm{J{\cdot}s}$$
$$= 4.136 \times 10^{-15}\ \mathrm{eV{\cdot}s}$$
$$c = 2.998 \times 10^{8}\ \mathrm{m/s}$$
$$G = 6.673 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$$
$$e = 1.602 \times 10^{-19}\ \mathrm{C}$$
$$m_e = 9.109 \times 10^{-31}\ \mathrm{kg}$$
$$m_p = 1.673 \times 10^{-27}\ \mathrm{kg}$$

Unit Conversions

$$1\ \mathrm{atm} = 1.013\ \mathrm{bar} = 1.013 \times 10^{5}\ \mathrm{N/m^2}$$
$$= 14.7\ \mathrm{lb/in^2} = 760\ \mathrm{mm\,Hg}$$
$$(T \text{ in } ^\circ\mathrm{C}) = (T \text{ in K}) - 273.15$$
$$(T \text{ in } ^\circ\mathrm{F}) = \tfrac{9}{5}(T \text{ in } ^\circ\mathrm{C}) + 32$$
$$1\ ^\circ\mathrm{R} = \tfrac{5}{9}\ \mathrm{K}$$
$$1\ \mathrm{cal} = 4.186\ \mathrm{J}$$
$$1\ \mathrm{Btu} = 1054\ \mathrm{J}$$
$$1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$$
$$1\ \mathrm{u} = 1.661 \times 10^{-27}\ \mathrm{kg}$$

PC2230, Midterm test makeup, 17 Mar 2022, 10:00-11:50 AM, closed book, each sub-question is 7.7 marks

Problem 1. Imagine some helium in a cylinder with an initial volume of 1 liter and an initial pressure of 1 atm. Somehow the helium is made to expand to a final volume of 3 liters, in such a way that its pressure rises in direct proportional to its volume.

(a) Sketch a graph of pressure vs. volume for this process.
(b) Calculate the work done on helium gas during this process, assuming that there are no "other" types of work done.
(c) Calculate the change in the helium's energy content during this process.
(d) Calculate the amount of heat added to or removed from the helium during this process.
(e) Describe what you might do to cause the pressure to rise as the helium expands.

Problem 2. Consider the Einstein solid with N oscillators and q units of energy.

(a) Give the formula for the multiplicity $\Omega(N, q)$.
(b) If we have four oscillators and two units of energy, what is the multiplicity? Representing each possible microstate as a series of dots (for the energy units) and vertical lines (for the separating walls), picture all the microstates of multiplicity $\Omega(4,2)$.
(c) Back to the general case of (a), derive the entropy of the Einstein solid S using the Sterling's approximation.
(d) Based on the result of (c), derive the energy of the Einstein solid U as a function of temperature T. Here we assume each energy units is ϵ, while q takes non-negative integers.

Problem 3. A heat pump is an electric device that heats a building by pumping heat in from the cold outside. In other words, it's the same as a refrigerator, but its purpose is to warm the hot reservoir rather than to cool the cold reservoir (even though it does both). Let us define the following standard symbols, all taken to be positive by convention.

$T_h =$ temperature inside building

$T_c =$ temperature outside

$Q_h =$ heat pumped into building in 1 day

$Q_c =$ heat taken from outdoors in 1 day

$W =$ electric energy used by heat pump in 1 day

(a) Explain why the "coefficient of performance" (COP) for the heat pump should be defined as Q_h/W.

(b) What relation among Q_h, Q_c, and W is implied by energy conservation alone? Will energy conservation permit the COP to be greater than 1?

(c) Use the second law of thermodynamics to derive an upper limit on the COP, in terms of the temperatures T_h and T_c alone.

(d) Explain why a heat pump is better than an electric furnace, which simply converts electric work directly into heat. (Include some numerical estimates).

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