SMA5202 Tech & Tools

Mathematica Assignment, Part II, due Friday, 18 July 2003

1. We will now consider a very simple model of population growth that involves two species (namely, Fox and Chicken). Let us denote the number of Fox and Chickens at (discrete) time i with F_i and C_i.

$$\begin{pmatrix} F_{i+1} \\ C_{i+1} \end{pmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ -k & 1.2 \end{bmatrix} \begin{pmatrix} F_i \\ C_i \end{pmatrix}.$$

If we examine the above equation, we will know that our model assumes that if there is no fox, the chicken population will increase with time. However, the chicken population will decrease with an increase in the fox population (determined by the kill rate k). This model also assumes that if there is no chicken, the fox population will decrease with time. However, the fox population will increase with an increase in the chicken population.

In this exercise, we will study the effect of different kill rates (k) on the population. The initial state of the population is assumed to be as follow: $F_0 = 100$, $C_0 = 1000$.

(a) For $k = 0.1$ and $k = 0.18$, compute the eigenvalue for the matrix and plot F_i and C_i for $i = 1$ to 100.

(b) Use Mathematica to find the value of k where the maximum eigenvalue is 1. For this value of k, plot F_i and C_i for $i = 1$ to 100.

Any conclusion you can draw of the relationship between eigenvalues and population growth (or growth rates)?

2. The Legendre polynomials, $P_n(x)$, appears in the solutions of Laplacian equation, $\nabla^2 f = 0$. There are many formulas related related to $P_n(x)$.

(a) Use the definition

$$g(t,x) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \quad |t| < 1,$$

find $P_n(x)$ for n from 0 to 5.

(b) Verify for n equal 0 to 5, that

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$

(c) Verify that for $n = 1$, 2, 3, and 4,

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$

and

$$(1 - x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0, \quad n = 0, 1, 2, \cdots$$

where prime $'$ denotes derivative with respect to x.

(d) And verify that (for n from 0 to 5)

$$P_n(x) = \frac{1}{2\pi}\int_0^{2\pi}(x + \sqrt{x^2-1}\cos\varphi)^n\,d\varphi,$$

(e) and that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}.$$

(f) And finally, show that $P_n(x)$ are orthogonal (for $n, m = 0$ to 5 only)

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{n,m}.$$