CZ3272 MC & MD, Tutorial 2 (*for week 4-5, Wednesday 6 or 13 Sep 06*)

1. Suppose that the probability for integer j from 1 to 10 is given by this table:

j	1	2	3	4	5	6	7	8	9	10
p_j	0.1	0.2	0.4	0.05	0.05	0.1	0.025	0.025	0.025	0.025

Design an algorithm to generate random integers j that are distributed according the above, using the idea of Pei Lucheng. Of course, the method should be efficient with minimum number of checks for conditions [No computer code is required].

2. Prove a formula. Given the linear congruential sequence defined by

$$X_{n+1} = (a X_n + c) \bmod m,$$

show that,

$$X_{n+k} = [\, a^k X_n + (a^k - 1)c/(a - 1)\,] \bmod m.$$

The equation expresses the $(n+k)$-th term directly in terms of the n-th term. It follows that the subsequence consisting of every k-th term of X_n is another linear congruential sequence having the multiplier $a^k \bmod m$ and the increment $(a^k - 1)c/(a - 1) \bmod m$.

3. Interesting fact. If the modulus $m = 2^e$, the low-order bits are much less random than the high-order bits. More precisely, the least significant bit is either constant or strictly alternating. The last two bits cannot have a period of more than 4; and the low-order four bits has a period of length 16 or less. To demonstrate this mathematically, let's define

$$Y_n = X_n \bmod d,$$

where d is a divisor of m, show that

$$Y_{n+1} = (aY_n + c) \bmod d.$$

That is Y_n is also a linear congruential sequence with modulus d, multiplier a, and increment c.

4. Sampling random variables. (a) A random variable ξ with a uniform probability distribution in the interval [0, 1) is given. Work out a method to generate a random integer $k \geq 0$ from ξ with the probability distribution

$$P_k = (1-\alpha)\alpha^k, \quad k = 0, 1, 2, 3, \ldots$$

where α is some constant satisfying $0 < \alpha < 1$.

(b) Find a transformation to generate a random variable x from a uniformly distributed random number ξ with the distribution

$$p(x) = \frac{c}{1 + x^2}, \quad 0 < x < \infty$$

What should be the value for c?

CZ3272 MC & MD, Lab 2 (*for week 5-6, September 2006*)

Due by Friday 22 September 2006

1. Write a Monte Carlo program to compute $S(n)$ defined by

$$S(n) = \int_{D(x_0)} dx_1 \int_{D(x_0,x_1)} dx_2 \cdots \int_{D(x_0,x_1,\ldots,x_{n-1})} dx_n$$

where $D(x_0)$ is a unit circle centered at x_0= (0,0), $D(x_0,x_1)$ is the union of circles centered at x_0 and x_1, etc. The integration variable x_i is two-dimensional. See the lecture note slides (set 4) for more details.

Compute $S(1)$, $S(2)$, $S(3)$, and $S(4)$ together with error estimates, using simple sampling Monte Carlo method and compare with exact results, if available.

Will you be able to do some of the integrals by hand? Try it.

[For comparison, the known answers are as follows:

$$S(1) = \pi,$$

$$S(2) = \left(4 + 3\sqrt{3}/\pi\right)\left(\frac{\pi}{2}\right)^2,$$

$$S(3) = \left(8 + 14\sqrt{3}/\pi + 44/\pi^2\right)\left(\frac{\pi}{2}\right)^3,$$

$$S(4) = 86.02824\left(\frac{\pi}{2}\right)^4.$$

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