Gluon State

Thursday, 12 January 2023 9:43 AM

The question is how to get the 8 gluon state, or at least why it looks the way it does. Let's start from phenomenology.

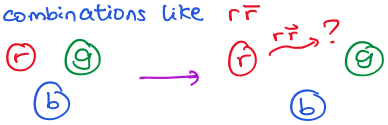
From the stable baryons made up of 3 quarks, we postulated that there are 3 colour charges, and it is stable when they are colourless.



The strong interaction involves an exchange of gluons, and the colour balance should be maintained. The simplest possible interaction model then involves a colour – anticolour pair. Consider:



The colour is changed, and the configuration remains stable or colourless. Naturally, for 3 colour charges, we expect $3\times3 = 9$ possibilities for a colour-anticolour pair. However, for combinations like $r\bar{r}$



There is no other quark to interact with, so there really are just $3\times2 = 6$ possibilities based on this consideration

$$b\bar{r} \quad r\bar{b} \quad r\bar{g}$$
$$g\bar{r} \quad g\bar{b} \quad b\bar{g}$$

These 6 states are independent of each other. To check that they are orthogonal, we need some representation.

Because the structure of the gluon state being a colour-anticolour pair, we can represent them by a 3×3 matrix:

$$\begin{array}{c} \\ r \\ b \\ g \end{array}\begin{array}{c} \begin{array}{ccc} \bar{r} & \bar{b} & \bar{g} \end{array} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{array}$$

So, for example:

$$b\bar{r} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad r\bar{b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For orthogonality in matrices, we check the trace of the matrix product:

$$\mathrm{tr}\{(b\bar{r})^{\dagger}(r\bar{b})\} = \mathrm{tr}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2\right\}$$
$$= \mathrm{tr}\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$
$$= 0$$

While these 6 states are orthogonal, they are not hermitian. To form hermitian matrices, notice that we can learn from the Pauli matrices, and form orthogonal hermitian states

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \sigma_x, \sigma_y \text{ for } r\bar{b}, b\bar{r}$$

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$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \rightarrow \sigma_x, \sigma_y \text{ for } r\bar{g}, g\bar{r}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \rightarrow \sigma_x, \sigma_y \text{ for } b\bar{g}, g\bar{b}$$

This accounts for the 6 gluon states

$$\frac{r\bar{b} + b\bar{r}}{\sqrt{2}}, \quad -i\,\frac{r\bar{b} - b\bar{r}}{\sqrt{2}}$$

$$\frac{r\bar{g} + g\bar{r}}{\sqrt{2}}, \quad -i\,\frac{r\bar{g} - g\bar{r}}{\sqrt{2}}$$

$$\frac{b\bar{g} + g\bar{b}}{\sqrt{2}}, \quad -i\,\frac{b\bar{g} - g\bar{b}}{\sqrt{2}}$$

PS: When constructing the matrix above, I ignored the coefficients for simplicity

PS: The numbering of λs are following the notation of Gell-Mann Matrices, that's why a number is skipped.

For the remaining states, since we used the idea of σ_x and σ_y to construct the states, it is natural to consider the σ_z equivalent:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

but these are not independent. For example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

However, we can retain one of them:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \lambda_3 \longrightarrow \frac{r\bar{r} - b\bar{b}}{\sqrt{2}}$$

Taking 2 of them also does not work because

$$\mathrm{tr}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$= \mathrm{tr}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= 1,$$

they are not orthogonal. To mimic the Pauli matrices, then we need another traceless hermitian matrix that is diagonal (because off diagonals are already covered by the earlier 6 gluon states) and orthogonal to λ_3:

Suppose $$\lambda_8 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

We have: $a + b + c = 0$

$$\mathrm{tr}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \right\} = 0$$

$$\mathrm{tr}\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = 0$$

$$a = b$$

$$\therefore \quad c = -2a$$

Then, $$\lambda_8 \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

where the last variable a is fixed by normalisation. This gives us the final gluon state:

$$\lambda_8 \rightarrow \frac{r\bar{r} + b\bar{b} - 2g\bar{g}}{\sqrt{6}}$$

But from Pauli matrices, we know that the identity makes the set complete, and that produces the 3×3 possibilities as was expected at the start. What about it?

Well, "λ_0" $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

as was expected at the start. What about it?

Well, "λ_9" $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

produces a state $\frac{r\bar{r} + b\bar{b} + g\bar{g}}{\sqrt{3}}$

that is symmetric under colour permutation, which means it is colourless. We know colourless state is stable, and if this gluon exists, then strong force would be long range. Since the strong force is not long range, then this last gluon state does not exist

PS: Because "λ_9" is a stable configuration much different from the other 8, we call this the colour singlet state (like the spin ground state) and the other 8 octet state (like the spin excited triplet).