Noether's Theorem

Tuesday, 31 January 2023 2:43 PM

Review of Lagrangian Mechanics

The conservation laws are closely related to symmetry. To understand that, let's start from classical mechanics and review the Lagrangian formalism.

Recall calculus of variation:
Suppose we want to optimise a functional

$$S = \int dt\, L(t, q(t), \dot{q}(t))$$

Then, around the stationary point:

$$0 = \delta S$$

$$= \int dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right)$$

We use integration by parts

$$\frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$$

$$= \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{\text{boundary}} + \int dt\, \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right]$$

For fixed boundary: $\delta q|_{\text{boundary}} = 0$

So, for $\delta S = 0$ for all choices of δq, we must have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$$

Turns out, Newton's equation can be cast as a problem for calculus of variation. In 1D, Newton's equation:

$$\frac{dp}{dt} = -\frac{\partial V}{\partial x}$$

where $V(x)$ is the potential energy for a conservative force

Then:

$$\frac{d}{dt}(m\dot{x}) = \frac{\partial}{\partial x}(-V)$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2 \right) \right) = \frac{\partial}{\partial x}(-V)$$

If we take

$$L = \frac{1}{2} m \dot{x}^2 - V(x), \quad \rightsquigarrow \text{Lagrangian}$$

then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

is the Newton's equation. We call

$$S[q(t), \dot{q}(t)] = \int dt\, L(t, q(t), \dot{q}(t))$$

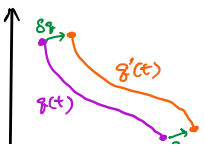
action. Since action is just a number, it does not matter what coordinate system we choose to represent L, so we speak of generalised coordinates $q(t)$.

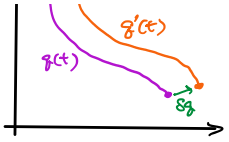
Noether's Theorem

Noether's theorem states that for every continuous symmetry, there is a corresponding conserved quantity. To view this physically, consider an optimised path $q(t)$. If we perturb the path slightly by δq, since $q(t)$ is optimised, we must have

$$\delta S = 0$$

around the optimised path.



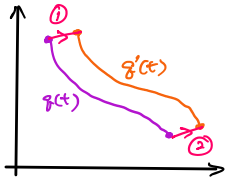


$\delta S = 0$

around the optimised path.

By symmetry, we mean that the transformation leaves the dynamics and system unchanged. This means that we must have

$$S[q(t)] = S[q'(t)]$$



But since

$$\delta S = 0 ,$$

we have

$$S[q(t)] = S_{①} + S[q'(t)] - S_{②}$$

↳ -ve because ② should go the other direction

$$\Rightarrow S_{①} = S_{②}$$

It follows that some quantity $S_{①} = S_{②}$ is conserved. Therefore, if there is a continuous symmetry, there is a conserved quantity. This is Noether's theorem

Pls: We talk of continuous symmetry because we investigated differential changes. This does not work for discrete symmetry.

If we consider the transformation as changing the beginning and end points of $q(t)$ in x, we see from above

$$\frac{\partial L}{\partial \dot{x}} \delta x \Big|_{\text{boundary}} = 0$$

Since $\frac{\partial L}{\partial \dot{x}} = p$ = momentum,

we have

$$p_{\text{initial}} = p_{\text{final}} .$$

Translation symmetry leads to conservation of momentum

Pls: We see that by translating the boundary points, we change the action by $p\,\delta x$. We speak of momentum p as a generator of translation. This is how we identified the momentum operator in QM

Some Examples

We have just seen that translation symmetry leads to conservation of momentum. We could also deduce it from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{dp}{dt} = 0 = \frac{\partial L}{\partial x} .$$

In general, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$,

a symmetry in generalised coordinates lead to conservation in the generalised momentum. For example, if q is rotational angle, then we have conservation of angular momentum.

What about time translation symmetry? Consider

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}$$

$$\frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) - \dot{q}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

$$= \frac{\partial L}{\partial t} + \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\dot{q} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

$\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] = 0$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = -\frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = -\frac{\partial L}{\partial t}$$

↳ This is because of convention

If L is independent on time t, i.e. time translation symmetry, then $\frac{\partial L}{\partial \dot{q}}\dot{q} - L$ is conserved.

If $L = \frac{1}{2}m\dot{x}^2 - V(x) \Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$$\therefore \frac{\partial L}{\partial \dot{x}}\dot{x} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x)$$

$$= \frac{1}{2}m\dot{x}^2 + V(x)$$

$$= \mathcal{H} \rightsquigarrow \text{Hamiltonian}$$

So, time translation symmetry implies that total energy is conserved.

Pls: Now you see why the convention. It is to make sure that the conserved quantity is the energy

Noether's theorem applies to QM too. In QM, suppose we have a symmetry transformation U that leaves the Hamiltonian $\mathcal{H}$ unchanged:

$$U^\dagger \mathcal{H} U = \mathcal{H}$$

$$\mathcal{H}U = U\mathcal{H} \Rightarrow [U, \mathcal{H}] = 0$$

Then, by Heisenberg equation of motion:

$$\frac{d}{dt}U = \underbrace{\frac{\partial U}{\partial t}}_{=0} + \underbrace{\frac{1}{i\hbar}[U, \mathcal{H}]}_{=0} = 0.$$

So, U is conserved.

Finally, in EM, we saw there is something called the gauge symmetry. the transformations

$$\vec{A} \mapsto \vec{A} + \vec{\nabla}\lambda \,, \quad V \mapsto V - \frac{\partial \lambda}{\partial t}$$

which leaves $\vec{E}$ and $\vec{B}$ field unchanged. This leads to conservation of charge via the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

but that is more mathematically involved and the derivation is beyond the scope of this module