Solutions

(1) $Z^2 = \dfrac{|1\rangle\langle 2|1\rangle\langle 2|}{\langle 2|1\rangle\langle 2|1\rangle} = \dfrac{|1\rangle\langle 2|}{\langle 2|1\rangle} = Z$ is immediate. The eigenvalues z of Z must therefore obey $z^2 = z$, so that $z=0$ and $z=1$ are the only possibilities. Indeed,

$$Z|1\rangle = \frac{|1\rangle\langle 2|1\rangle}{\langle 2|1\rangle} = |1\rangle ,$$

$$\langle 2|Z = \frac{\langle 2|1\rangle\langle 2|}{\langle 2|1\rangle} = \langle 2|$$

show what is reasonably obvious in the first place, namely that $|1\rangle$ is the eigenket of Z to eigenvalue 1, and $\langle 2|$ is the eigenbra of Z. Every ket $|\;\rangle$ with $\langle 2|\;\rangle = 0$ is eigenket to eigenvalue 0, and every bra $\langle\;|$ with $\langle\;|1\rangle$ is eigenbra to eigenvalue 0.

(2) Recall the definitions $\frac{\partial}{\partial X} f = \frac{1}{i\hbar}[f, P]$, $\frac{\partial}{\partial P} f = \frac{1}{i\hbar}[X, f]$ to express the difference as

$$\frac{\partial}{\partial X}\frac{\partial}{\partial P} f - \frac{\partial}{\partial P}\frac{\partial}{\partial X} f = \left(\frac{1}{i\hbar}\right)^2 \big[[X,f],P\big] - \left(\frac{1}{i\hbar}\right)^2 \big[X,[f,P]\big]$$

$$= \left(\frac{1}{i\hbar}\right)^2 \Big(\big[[X,f],P\big] + \big[[f,P],X\big]\Big)$$

$$= \left(\frac{1}{i\hbar}\right)^2 \Big(\underbrace{\big[[X,f],P\big] + \big[[f,P],X\big] + \big[[P,X],f\big]}_{=0 \text{ as a consequence of the Jacobi identity}} + \underbrace{\big[[X,P],f\big]}_{=0 \text{ because } [X,P]=i\hbar \text{ commutes with } f}\Big)$$

$$= 0 .$$

Conclusion: $\frac{\partial}{\partial X}\frac{\partial}{\partial P} f = \frac{\partial}{\partial P}\frac{\partial}{\partial X} f$; they are the same.

(3) We have $P(t) = P(0)$ and $X(t) = X(0) + \frac{t}{M} P(0)$, so that

$$P(t)^2 = P(0)^2, \quad X(t)^2 = X(0)^2 + \frac{t^2}{M^2} P(0)^2 + \frac{t}{M}\left(X(0)P(0) + P(0)X(0)\right)$$

and

$$\langle P(t) \rangle = \langle P(0) \rangle = 0,$$

$$\langle P(t)^2 \rangle = \langle P(0)^2 \rangle = p_0^2,$$

$$\langle X(t) \rangle = \langle X(0) \rangle + \frac{t}{M} \langle P(0) \rangle = 0,$$

$$\langle X(t)^2 \rangle = \langle X(0)^2 \rangle + \frac{t^2}{M^2} \langle P(0)^2 \rangle + \frac{t}{M} \langle (X(0)P(0) + P(0)X(0)) \rangle$$
$$= x_0^2 + \frac{t^2}{M^2} p_0^2$$

follow, so that

$$\delta P(t) = \sqrt{\langle P(t)^2 \rangle - \langle P(t) \rangle^2} = p_0,$$

$$\delta X(t) = \sqrt{\langle X(t) \rangle^2 - \langle X(t) \rangle^2} = \sqrt{x_0^2 + \left(\frac{p_0 t}{M}\right)^2}.$$

The uncertainty relation is, of course, obeyed at all times

$$\delta X(t)\, \delta P(t) = x_0 p_0 \sqrt{1 + \left(\frac{p_0 t}{M x_0}\right)^2} \geq x_0 p_0 \geq \frac{\hbar}{2}.$$

(4) Write $A^{+n} A^n = f_n(A^+A)$, then $f_0(A^+A) = 1$ and $f_1(A^+A) = A^+A$. Further we have

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so that we get the recurrence relation

$$f_n(A^+A) = A^+A\, f_{n-1}(A^+A-1).$$

After looking at the first few

$$f_2(A^+A) = A^+A(A^+A-1),$$
$$f_3(A^+A) = A^+A(A^+A-1)(A^+A-2)$$

we recognize the pattern and infer that

$$f_n(A^+A) = A^+A(A^+A-1)(A^+A-2)\cdots(A^+A-n+1)$$
$$= \prod_{k=0}^{n-1}(A^+A-k) = \frac{(A^+A)!}{(A^+A-n)!}.$$

One checks easily that the recurrence relation is obeyed.

(5) We differentiate with respect to the numerical parameter λ and get

$$\frac{\partial}{\partial\lambda} e^{-i\lambda S/\hbar}\vec{R}e^{i\lambda S/\hbar} = e^{-i\lambda S/\hbar}\frac{1}{i\hbar}\underbrace{[S,\vec{R}]}_{=-i\hbar\vec{R}}e^{i\lambda S/\hbar},$$

$$= -e^{-i\lambda S/\hbar}\vec{R}e^{i\lambda S/\hbar}.$$

This differential equation is of the simple form $f'(\lambda)=-f(\lambda)$, solved by $f(\lambda)=e^{-\lambda}f(0)$, so that

$$e^{-i\lambda S/\hbar}\vec{R}e^{i\lambda S/\hbar} = e^{-\lambda}\vec{R}.$$

Likewise one gets $e^{-i\lambda S/\hbar}\vec{P}e^{i\lambda S/\hbar} = e^{\lambda}\vec{P}$.

(6) Suppressing any other potential quantum numbers, we have $|\,\rangle = |l,m\rangle$, $\vec{L}^2|l,m\rangle = |l,m\rangle \hbar^2 l(l+1)$, $L_3|l,m\rangle = |l,m\rangle \hbar m$
and

$$(L_1 + iL_2)\,|l,m\rangle \propto |l,m+1\rangle \quad \text{(or vanishes if } m=l\text{)} ,$$

$$(L_1 + iL_2)^2\,|l,m\rangle \propto |l,m+2\rangle \quad \text{(or vanishes if } m\geq l-1\text{)} .$$

Since $|l,m\rangle$ is orthogonal to $|l,m+1\rangle$ and $|l,m+2\rangle$, we thus have

$$\langle (L_1+iL_2)\rangle = 0 \quad ,$$
$$\langle (L_1+iL_2)^2\rangle = 0$$

or, after separating the real and imaginary parts,

$$\langle L_1\rangle = \langle L_2\rangle = 0 \quad ,$$
$$\langle (L_1^2 - L_2^2)\rangle = 0 \quad , \quad \langle (L_1L_2 + L_2L_1)\rangle = 0 .$$

With $\langle (L_1^2+L_2^2)\rangle = \langle (\vec{L}^2 - L_3^2)\rangle = \hbar^2\left(l(l+1) - m^2\right)$
this gives

$$\langle L_1^2\rangle = \langle L_2^2\rangle = \frac{\hbar^2}{2}\left(l(l+1) - m^2\right)$$

and then

$$\delta L_1 = \delta L_2 = \frac{\hbar}{\sqrt{2}}\sqrt{l(l+1) - m^2} .$$

Since $[L_1, L_2] = i\hbar L_3$, the uncertainty relation for L_1 and L_2 is

$$\delta L_1 \delta L_2 \geq \frac{1}{2}\left|\langle i[L_1,L_2]\rangle\right| = \frac{\hbar}{2}\left|\langle L_3\rangle\right| ,$$

here:

$$\frac{\hbar^2}{2}\left(l(l+1) - m^2\right) \geq \frac{\hbar^2}{2}\, m ,$$

which is tantamount to $l \geq |m|$ and thus obeyed, indeed.