<u>Solutions</u>

(1) Simply $[A^2, A^{+4}] = A[A, A^{+4}] + [A, A^{+4}]A$

$$= A\,\frac{\partial A^{+4}}{\partial A^{+}} + \frac{\partial A^{+4}}{\partial A^{+}}\,A = 4\,A A^{+3} + 4 A^{+3} A$$

$$= 8\,A^{+3}A + 4\,[A, A^{+3}] = 8A^{+3}A + 12A^{+2}.$$

(2) Consider, for instance, the first component of the first statement,

$$(\vec{R}\times\vec{L} + \vec{L}\times\vec{R})_1 = R_2L_3 - R_3L_2 + L_2R_3 - L_3R_2$$

$$= [R_2, L_3] + [L_2, R_3] = i\hbar R_1 + i\hbar R_1 = 2i\hbar R_1,$$

and likewise for the second and third component. The argument for the second statement is fully analogous.

(3a) We recall

$$(L_1 \pm i L_2)\,|\ell, m\rangle = |\ell, m\pm 1\rangle\,\hbar\sqrt{(\ell\mp m)(\ell\pm m+1)}$$

so that, taking half the sum,

$$L_1\,|\ell, m\rangle = |\ell, m+1\rangle\,\tfrac{1}{2}\hbar\sqrt{(\ell-m)(\ell+m+1)}$$

$$+\,|\ell, m-1\rangle\,\tfrac{1}{2}\hbar\sqrt{(\ell+m)\,\ell-m+1)}\,.$$

For $\ell=2$, in particular,

$$L_1\,|2,2\rangle = |2,1\rangle\,\tfrac{1}{2}\hbar\sqrt{(2+2)(2-2+1)} = |2,1\rangle\hbar\,,$$

$$L_1\,|2,-2\rangle = |2,-1\rangle\,\tfrac{1}{2}\hbar\sqrt{(2+2)(2-2+1)} = |2,-1\rangle\hbar\,,$$

as well as

$$L_1|2,1\rangle = |2,2\rangle \tfrac{1}{2}\hbar\sqrt{(2-1)(2+1+1)} + |2,0\rangle \tfrac{1}{2}\hbar\sqrt{(2+1)(2-1+1)}$$

$$= |2,2\rangle \hbar + |2,0\rangle \hbar\sqrt{3/2},$$

$$L_1|2,-1\rangle = |2,-2\rangle \tfrac{1}{2}\hbar\sqrt{(2-1)(2+1+1)} + |2,0\rangle \tfrac{1}{2}\hbar\sqrt{(2+1)(2-1+1)}$$

$$= |2,-2\rangle \hbar + |2,0\rangle \hbar\sqrt{3/2},$$

and

$$L_1|2,0\rangle = |2,1\rangle \tfrac{1}{2}\hbar\sqrt{(2-0)(2+0+1)} + |2,-1\rangle \tfrac{1}{2}\hbar\sqrt{(2+0)(2-0+1)}$$

$$= \big(|2,1\rangle + |2,-1\rangle\big)\hbar\sqrt{3/2}.$$

(3b) We thus have

$$L_1\big(|2,2\rangle\alpha - |2,0\rangle\beta + |2,-2\rangle\alpha\big)$$

$$= \big(|2,1\rangle + |2,-1\rangle\big)\hbar\big(\alpha - \beta\sqrt{3/2}\big) = 0$$

so that

$$\alpha = \sqrt{\frac{3}{2}}\,\beta$$

and $2|\alpha|^2 + |\beta|^2 = 1$ requires $4|\beta|^2 = 1$. We choose

$\beta = \frac{1}{2}$ and then get $\alpha = \sqrt{\frac{3}{8}}$.

(4a) We have $\frac{d}{dx}\psi(x) = -K^2 x\,\psi(x)$ and so get

211 $$\langle P^2\rangle = \int dx\,\left|\hbar\frac{d}{dx}\psi(x)\right|^2 = \hbar^2 K^5 \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\,x^2 e^{-K^2x^2}$$

$$= \hbar^2 \kappa^5 \frac{1}{\sqrt{\pi}} \left(-\frac{1}{2\kappa}\frac{\partial}{\partial \kappa}\right) \underbrace{\int dx\, e^{-\kappa^2 x^2}}_{= \frac{\sqrt{\pi}}{\kappa}} = \frac{\hbar^2 \kappa^2}{2} .$$

Further,

$$\langle |X|^3 \rangle = \int dx\, |x|^3 |\psi(x)|^2 = 2\int_0^\infty dx\, x^3 \frac{\kappa}{\sqrt{\pi}} e^{-\kappa^2 x^2}$$

$$y = x^2 \quad \downarrow \quad = \frac{\kappa}{\sqrt{\pi}} \underbrace{\int_0^\infty dy\, y\, e^{-\kappa^2 y}}_{\frac{1}{\kappa^4}} = \frac{1}{\sqrt{\pi}\kappa^3} .$$

(4b) We have

$$E_0 \leq \frac{1}{2M} \langle P^2 \rangle + \lambda^2 \langle |X|^3 \rangle = \frac{\hbar^2 \kappa^2}{4M} + \frac{\lambda^2}{\sqrt{\pi}\kappa^3}$$

for all $\kappa > 0$, so that we get the best upper bound of this kind for the κ value that minimizes the right-hand side. We find it by differentiation:

$$\frac{\hbar^2 \kappa}{2M} - \frac{3\lambda^2}{\sqrt{\pi}\kappa^4} = 0 \quad \text{or} \quad \kappa^5 = \frac{6}{\sqrt{\pi}} \frac{\lambda^2 M}{\hbar^2}$$

and obtain

$$E_0 \leq \left[\frac{1}{4}\left(\frac{6}{\sqrt{\pi}}\right)^{2/5} + \frac{1}{\sqrt{\pi}}\left(\frac{6}{\sqrt{\pi}}\right)^{-3/5}\right] \left(\frac{\hbar^2}{M}\right)^{3/5} \lambda^{4/5}$$

$$= 5 \cdot 2^{-8/5} \cdot 3^{-3/5} \cdot \pi^{-1/5} \left(\frac{\hbar^2}{M}\right)^{3/5} \lambda^{4/5}$$

$$= \frac{5}{(6912\pi)^{1/5}} \left(\frac{\hbar^6 \lambda^4}{M^3}\right)^{1/5} .$$

(5a) We have

$$H = \underbrace{\hbar\omega\, A^+A}_{H_0} \underbrace{- F\frac{\ell}{\sqrt{2}}(A+A^+)}_{H_1} \quad \text{with } \ell = \sqrt{\frac{\hbar}{M\omega}}$$

and the unperturbed states $|n^{(0)}\rangle$ are the Fock states, the eigenstates of A^+A. The relevant matrix elements of H_1 are

$$\langle m^{(0)}|H_1|n^{(0)}\rangle \quad \text{for } n=0,$$

that is

$$\langle m^{(0)}|\left(-\frac{F\ell}{\sqrt{2}}\right)(A^+ + A)|0^{(0)}\rangle = -\frac{F\ell}{\sqrt{2}}\delta_{m,1},$$

since $A|0^{(0)}\rangle = 0$ and $A^+|0^{(0)}\rangle = |1^{(0)}\rangle$.

Accordingly, the 1st order change vanishes

$$E_0^{(1)} = \langle 0^{(0)}|H_1|0^{(0)}\rangle = 0,$$

and the 2nd order change is

$$E_0^{(2)} = -\sum_{m\neq 0} \frac{|\langle m^{(0)}|H_1|0^{(0)}\rangle|^2}{E_m^{(0)} - E_0^{(0)}} = -\frac{(F\ell/\sqrt{2})^2}{\hbar\omega}$$

$$= -\frac{1}{2}\frac{F^2\ell^2}{\hbar\omega} = -\frac{1}{2}\frac{F^2}{M\omega^2}$$

where the unperturbed energies $E_m^{(0)} = \hbar\omega m$ are taken into due account. Thus to 2nd order in F we have

$$E_0 = E_0^{(0)} + E_0^{(1)} + E_0^{(2)} = -\frac{1}{2}\frac{F^2}{M\omega^2}.$$

(5b) We complete the square in H,

$$H = \frac{1}{2M} P^2 + \frac{1}{2} M\omega^2 \left(X - \frac{F}{M\omega^2}\right)^2 - \frac{F^2}{2M\omega^2} - \frac{1}{2}\hbar\omega$$

$$\equiv \overline{H} - \frac{F^2}{2M\omega^2}$$

where

$$\overline{H} = \frac{1}{2M} P^2 + \frac{1}{2} M\omega^2 \left(X - \frac{F}{M\omega^2}\right)^2 - \frac{1}{2}\hbar\omega$$

is the Hamilton operator of a harmonic oscillator located at $x = \frac{F}{M\omega^2}$ rather than $x = 0$. This change in location has, of course, no effect on the eigenvalues, so that $\overline{H}$ has the same eigenvalues as $H_0 = \frac{1}{2M} P^2 + \frac{1}{2} M\omega^2 X^2 - \frac{1}{2}\hbar\omega$. In particular the ground state of $\overline{H}$ has energy 0. Accordingly, the exact ground state energy of H is

$$E_0 = -\frac{F^2}{2M\omega^2},$$

which is <u>identical</u> with the 2nd-order perturbation result of (5a).