Problem 1 (10 points)
Ladder operators A, $A^\dagger$ of a one-dimensional harmonic oscillator: Write the commutator $[A^2, {A^\dagger}^4]$ in its $A^\dagger, A$-ordered form.

Problem 2 (15 points)
For the vector operators $\vec{R}$, $\vec{P}$, and $\vec{L}$ of position, momentum, and orbital angular momentum, respectively, show that the statements

$$\vec{R}\times\vec{L}+\vec{L}\times\vec{R}=2\mathrm{i}\hbar\vec{R} \qquad \text{and} \qquad \vec{P}\times\vec{L}+\vec{L}\times\vec{P}=2\mathrm{i}\hbar\vec{P}$$

hold.

Problem 3 (25 points)
Orbital angular momentum: vector operator $\vec{L}$ with components L_1,L_2, and L_3. Denote by $|l,m\rangle$ the joint eigenkets of $\vec{L}^2$ and L_3 (eigenvalue $\hbar^2 l(l+1)$ of $\vec{L}^2$; eigenvalue $\hbar m$ of L_3).

(a) Exploit the known results of applying $L_1 \pm \mathrm{i}L_2$ to $|l,m\rangle$ to find $L_1|l,m\rangle$ for $l=2$ and $m=0,\pm1,\pm2$.

(b) Determine the coefficients α and β in

$$L_1\Big(|l=2,m=2\rangle\alpha-|l=2,m=0\rangle\beta+|l=2,m=-2\rangle\alpha\Big)=0$$

such that this equation holds and $2|\alpha|^2+|\beta|^2=1$.

Problem 4 (25 points)
Consider the one-dimensional Hamilton operator

$$H=\frac{1}{2M}P^2+\lambda^2|X|^3\,,$$

where X is the particle's position operator, P is its momentum operator, M is its mass, and $\lambda>0$ determines the strength of the cubic potential.

(a) Determine the expectation values of P^2 and $|X|^3$ in a state with a Gaussian wave function $\psi(x)=\langle x|\ \rangle=\pi^{-1/4}\sqrt{\kappa}\,\mathrm{e}^{-\frac{1}{2}\kappa^2x^2}$ (with $\kappa>0$.).

(b) Use them to get an upper bound on the ground-state energy E_0. [Hint: Remember the Rayleigh–Ritz variational principle; optimize the value of κ.]

Problem 5 (25 points)
The Hamilton operator

$$H=\frac{1}{2M}P^2+\frac{1}{2}M\omega^2X^2-\frac{1}{2}\hbar\omega-FX\,,$$

is that of a one-dimensional harmonic oscillator (position operator X, momentum operator P, circular frequency ω), perturbed by a force of constant strength F.

(a) Find the change in the ground-state energy to second order in F. [Hint: It may help to remember about ladder operators.]

(b) Determine the exact ground-state energy, and compare it with your result from part (a). [Hint: Complete a square.]