Solutions

(1a) The Heisenberg equations of motion are here

$$\frac{d}{dt} P = -\frac{\partial H}{\partial X} = -\gamma P \quad , \quad \frac{d}{dt} X = \frac{\partial H}{\partial P} = \frac{1}{M} P + \gamma X$$

and are solved by

$$P(t) = e^{-\gamma T} P(t_0) ,$$

$$X(t) = e^{\gamma T} X(t_0) + \frac{e^{\gamma T} - e^{-\gamma T}}{2\gamma M} P(t_0)$$

with $T \equiv t - t_0$.

(1b) We have

$$[X(t), P(t_0)] = [e^{\gamma T} X(t_0), P(t_0)] = i\hbar e^{\gamma T} .$$

(1c) Proceeding from

$$i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \langle x, t | H_t | p, t_0 \rangle$$

we express H_t in terms of $X(t)$ and $P(t_0)$. This gives

$$H = \frac{1}{2M} e^{-2\gamma T} P(t_0)^2 + \frac{1}{2}\gamma e^{-\gamma T}\left(X(t) P(t_0) + P(t_0) X(t)\right)$$

$$= \frac{1}{2M} e^{-2\gamma T} P(t_0)^2 + \gamma e^{-\gamma T} X(t) P(t_0) - \frac{1}{2} i\hbar\gamma$$

so that

$$i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \left(\frac{1}{2M} e^{-2\gamma T} p^2 + \gamma e^{-\gamma T} xp - \frac{1}{2} i\hbar\gamma\right) \langle x, t | p, t_0 \rangle$$

$$= \langle x, t | p, t_0 \rangle \frac{\partial}{\partial t} \left(\frac{p^2}{2M} \frac{1 - e^{-2\gamma T}}{2\gamma} + xp(1 - e^{-\gamma T}) - \frac{1}{2} i\hbar\gamma T\right) .$$

As a consequence, we get

$$\langle x,t|p,t_0\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-\gamma T/2}\, e^{ixpe^{-\gamma T}/\hbar}\, e^{-i\frac{p^2}{2M}\frac{1-e^{-2\gamma T}}{2\hbar\gamma}}$$

upon incorporating the initial condition

$$\langle x,t_0|p,t_0\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ixp/\hbar}.$$

(2a) For $H = H_0 + H_1$ with $H_0 = \hbar\omega A^\dagger A$ and $H_1 = \hbar\Omega(A^{\dagger 2} + A^2)$, the unperturbed states $|n^{(0)}\rangle$ are the standard Fock states. The relevant matrix elements of H_1 are those between the ground state $|0^{(0)}\rangle$ and any excited state $|m^{(0)}\rangle$,

$$\langle m^{(0)}|H_1|0^{(0)}\rangle = \hbar\Omega\langle m^{(0)}|A^{\dagger 2}|0^{(0)}\rangle$$

$$= \hbar\Omega\sqrt{2}\,\delta_{m,2}.$$

Accordingly, the first-order correction vanishes,

$$E_0^{(1)} = \langle 0^{(0)}|H_1|0^{(0)}\rangle,$$

and the second-order correction is

$$E_0^{(2)} = -\sum_{m=1}^{\infty}\frac{|\langle m^{(0)}|H_1|0^{(0)}\rangle|^2}{E_m^{(0)} - E_0^{(0)}}$$

$$= -\sum_{m=1}^{\infty}\frac{(\hbar\Omega\sqrt{2}\,\delta_{m,2})^2}{m\hbar\omega} = -\hbar\frac{\Omega^2}{\omega}.$$

(2b) We have (pages 57-59 of the notes)

$$A^{\dagger 2} + A^2 = \frac{M\omega}{\hbar}X^2 - \frac{1}{\hbar M\omega}P^2$$

and

$$\hbar\omega A^\dagger A = \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2X^2 - \frac{1}{2}\hbar\omega,$$

so that

$$H = \left(\frac{1}{2M} - \frac{\Omega}{M\omega}\right)P^2 + \left(\frac{1}{2}M\omega^2 + M\Omega\omega\right)X^2 - \frac{1}{2}\hbar\omega$$

$$= \frac{1}{2\bar{M}}P^2 + \frac{1}{2}\bar{M}\bar{\omega}^2 X^2 - \frac{1}{2}\hbar\omega$$

which is a harmonic oscillator with mass $\bar{M}$ and frequency $\bar{\omega}$ that are given by

$$\bar{M} = \left(\frac{1}{M} - \frac{2\Omega}{M\omega}\right)^{-1}, \quad \bar{M}\bar{\omega}^2 = M\omega^2 + 2M\Omega\omega .$$

Thus,

$$\bar{\omega} = \sqrt{\omega^2 - 4\Omega^2}$$

and the true ground-state energy is

$$E_0 = \frac{1}{2}\hbar\bar{\omega} - \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\omega\left(\sqrt{1 - \frac{4\Omega^2}{\omega^2}} - 1\right)$$

$$= \frac{1}{2}\hbar\omega\left(-\frac{2\Omega^2}{\omega^2} + O\left((\Omega/\omega)^4\right)\right)$$

$$\simeq -\frac{\hbar\Omega^2}{\omega} \quad \text{to second order in } \Omega.$$

This agrees with the second-order result in (2b), as it should.

(2c) Since $\bar{\omega}^2 = \omega^2 - 4\Omega^2$, reasonable values of Ω are in the range

$$-\frac{1}{2}\omega < \Omega < \frac{1}{2}\omega.$$

(3a) One has

$$\langle P^2\rangle = \hbar^2\int dx\,\left|\frac{d}{dx}\psi(x)\right|^2 = \hbar^2\int dx\left(-\kappa^2 x\,\psi(x)\right)^2$$

$$= \frac{\hbar^2}{\sqrt{\pi}}\,\kappa^5\int dx\; x^2 e^{-\kappa^2x^2} = \frac{1}{2}\hbar^2\kappa^2 ,$$

and

$$\langle X^4\rangle = \int dx\; x^4\,|\psi(x)|^2 = \frac{\kappa}{\sqrt{\pi}}\int dx\; x^4 e^{-\kappa^2x^2}$$

$$= \frac{3}{4\kappa^4} ,$$

after using $\int dx\, e^{-\kappa^2x^2} = \frac{\sqrt{\pi}}{\kappa}$ and $x^2 e^{-\kappa^2x^2} = -\frac{1}{2\kappa}\frac{\partial}{\partial\kappa}e^{-\kappa^2x^2}$ repeatedly.

(3b) According to the Rayleigh-Ritz variational principle,

$$E_0 \le \langle H\rangle = \frac{\hbar^2\kappa^2}{4M} + \frac{3\lambda^2}{4\kappa^4}$$

for all κ. The best upper bound of this kind obtains for the κ value that minimizes the right-hand side. We differentiate (simplest: with respect to κ^2) to determine it as the solution of

$$\frac{\hbar^2}{4M} - \frac{3\lambda^2}{2\kappa^6} = 0 , \text{ thus } \kappa^2 = \left(6\frac{\lambda^2 M}{\hbar^2}\right)^{1/3}$$

and

$$E_0 \le \left(\frac{3}{4}\right)^{4/3}\left(\frac{\hbar^2\lambda}{M}\right)^{2/3} .$$

(3c) Here we have

$$\langle P^2\rangle = \frac{3}{2}\hbar^2\kappa^2 \text{ and } \langle X^4\rangle = \frac{15}{4\kappa^4}$$

so that

$$E_1 \le \frac{3}{4}\frac{\hbar^2\kappa^2}{M} + \frac{15}{4}\frac{\lambda^2}{\kappa^4}$$

where the right-hand side is minimal for

$$\kappa^2 = \left(10\,\frac{\lambda^2 M}{\hbar^2}\right)^{1/3}$$

and

$$E_1 \le \frac{9}{8}(10)^{1/3}\left(\frac{\hbar^2\lambda}{M}\right)^{2/3}$$

obtains as the best upper bound of this kind on E_1.

(4a) Recall that $(L_1 \pm i L_2) f(L_3) = f(L_3 \mp \hbar)(L_1 \pm i L_2)$, so that

$$e^{-i\varphi L_3/\hbar}(L_1 \pm i L_2)\, e^{i\varphi L_3/\hbar} = e^{\mp i\varphi}(L_1 \pm i L_2)$$

and

$$e^{-i\varphi L_3/\hbar} L_1\, e^{i\varphi L_3/\hbar} = L_1\cos\varphi + L_2\sin\varphi,$$

$$e^{-i\varphi L_3/\hbar} L_2\, e^{i\varphi L_3/\hbar} = L_2\cos\varphi - L_1\sin\varphi$$

follow. Alternatively, we can differentiate with respect to φ and make use of $[L_1, L_3] = -i\hbar L_2$ and $[L_2, L_3] = i\hbar L_1$.

(4b) For $\varphi = \pi$, we have $L_1 e^{i\pi L_3/\hbar} = -e^{i\pi L_3/\hbar} L_1$, so that

$$L_1\, e^{i\pi L_3/\hbar}|l, m_1\rangle = e^{i\pi L_3/\hbar}|l, m_1\rangle(-m_1\hbar),$$

which states that

$$e^{i\pi L_3/\hbar}|l, m_1\rangle$$

is eigenket of L_1 with eigenvalue $-\hbar m_1$. It must therefore be equal to $|l, -m_1\rangle$ up to a phase factor. Accordingly,

$$|\langle l, m_1 | l, m_3\rangle| = |\langle l, -m_1 | l, m_3\rangle|.$$

It follows by symmetry that, quite generally, the moduli $|\langle l, m_1 | l, m_3\rangle|$ do not depend on the signs of m_1 or m_3.

(4c) In view of the symmetry established in (4b), the four probabilities for $m_1 = \pm 1$ and $m_3 = \pm 1$ are the same. Also the four probabilities for $m_1 = 0$, $m_3 = \pm 1$ and $m_1 = \pm 1$, $m_3 = 0$ are identical. To find the $m_1 = 0$, $m_3 = 0$ probability, now consider

$$L_1 \, |l=1, m_3=1\rangle =$$

$$= \left[\tfrac{1}{2}(L_1 + iL_2) + \tfrac{1}{2}(L_1 - iL_2)\right] |l=1, m_3=1\rangle = |l=1, m_3=0\rangle \frac{\hbar}{\sqrt{2}}$$

and $$L_1 \, |l=1, m_1=-1\rangle$$

$$= \left[\tfrac{1}{2}(L_1 + iL_2) + \tfrac{1}{2}(L_1 - iL_2)\right] |l=1, m_3=-1\rangle = |l=1, m_3=0\rangle \frac{\hbar}{\sqrt{2}},$$

or, after taking the difference,

$$L_1 \left[|l=1, m_3=1\rangle - |l=1, m_3=-1\rangle \right] = 0 .$$

This states that $|l=1, m_1=0\rangle$ is proportional to this difference and, therefore, we find

$$\langle l=1, m_1=0 | l=1, m_3=0\rangle = 0.$$

This is the central entry in the table

	$\ell=1$	m_3: +1	0	-1
m_1	+1	1/4	1/2	1/4
	0	1/2	0	1/2
	-1	1/4	1/2	1/4

Table of $|\langle \ell=1, m_1 | \ell=1, m_3\rangle|^2$ for $m_1, m_3 = 0, \pm 1$.

and all other entries are uniquely determined by the symmetry stated above and the normalization of each row and each column to unit sum.

THE END