Problem 1

Consider a particle in one-dimensional motion governed by the Hamilton operator

$$H = \frac{1}{2M}P^2 + \frac{1}{2}\gamma(XP + PX)\,,$$

where $X(t)$ is the particle's position operator, $P(t)$ is its momentum operator, M is its mass, and γ is a constant parameter.

(a) Solve the Heisenberg equations of motion, that is: state $X(t)$ and $P(t)$ in terms of $X(t_0)$ and $P(t_0)$. (10 points)

(b) Find the commutator $[X(t), P(t_0)]$. (5 points)

(c) Find the time transformation function $\langle x, t | p, t_0 \rangle$. [Hint: Use the results of parts (a) and (b).] (10 points)

Problem 2

The Hamilton operator

$$H = \hbar\omega A^\dagger A + \hbar\Omega({A^\dagger}^2 + A^2)$$

is that of a one-dimensional harmonic oscillator (ladder operators $A^\dagger$ and A, circular frequency ω), with a perturbation proportional to ${A^\dagger}^2 + A^2$ of strength Ω.

(a) Find the change in the ground-state energy to second order in Ω. (10 points)

(b) Determine the exact ground-state energy, and compare it with your result from part (a). [Hint: Express H in terms of position X and momentum P.] (10 points)

(c) In which range are the reasonable values of Ω? (5 points)

Problem 3

Consider the one-dimensional Hamilton operator

$$H = \frac{1}{2M}P^2 + \lambda^2 X^4\,,$$

where X is the particle's position operator, P is its momentum operator, M is its mass, and $\lambda > 0$ determines the strength of the quartic potential.

(a) Determine the expectation values $\langle P^2 \rangle$ and $\langle X^4 \rangle$ in a state with a Gaussian wave function $\psi(x) = \langle x|\ \rangle = \pi^{-1/4}\sqrt{\kappa}\,\mathrm{e}^{-\frac{1}{2}\kappa^2 x^2}$ (with $\kappa > 0$.). (8 points)

(b) Use them to get an upper bound on the ground-state energy E_0. [Hint: Remember the Rayleigh–Ritz variational principle; optimize the value of κ.] (8 points)

(c) Now use $\psi(x) = \pi^{-1/4}\sqrt{2\kappa^3}\,x\mathrm{e}^{-\frac{1}{2}\kappa^2 x^2}$ to find a good upper bound on the energy of the first excited state. (9 points)

Problem 4

Orbital angular momentum: vector operator $\vec{L}$ with components L_1, L_2, and L_3. Denote by $|l, m_j\rangle$ the joint eigenkets of $\vec{L}^2$ and L_j (eigenvalue $\hbar^2 l(l+1)$ of $\vec{L}^2$; eigenvalue $\hbar m_j$ of L_j).

(a) Find the effect on L_1 and L_2 of the unitary transformation associated with L_3, that is: evaluate

$$\mathrm{e}^{-\mathrm{i}\varphi L_3/\hbar} L_1\, \mathrm{e}^{\mathrm{i}\varphi L_3/\hbar} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}\varphi L_3/\hbar} L_2\, \mathrm{e}^{\mathrm{i}\varphi L_3/\hbar}\,.$$

[Hint: Differentiate or, more simply, recall some ladder-operator properties.] (10 points)

(b) Use the result of (a) for $\varphi = \pi$ to demonstrate that $\langle l, m_1 = 1|l, m_3\rangle$ and $\langle l, m_1 = -1|l, m_3\rangle$, for instance, differ only by a phase factor. Then conclude that transition probabilities such as $|\langle l, m_1|l, m_3\rangle|^2$ do not depend on the signs of the quantum numbers m_1 and m_3. (5 points)

(c) Determine the $3 \times 3 = 9$ transition probabilities $|\langle l = 1, m_1|l = 1, m_3\rangle|^2$, with $m_1, m_3 = 0, \pm 1$, between the three L_1 states and the three L_3 states to $l = 1$. [Hint: It's enough to calculate one of the nine numbers, the others can then be inferred by a symmetry argument that exploits the findings of part (b).] (10 points)