Question EXAM / 1

Write answers on this side of the paper only.

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$\boxed{S1}$ 50% of $\uparrow_x$ & 50% of $\downarrow_z$ $\xrightarrow[\uparrow_x \text{ selector}]{}$ 50% $\uparrow_x$ & 25% $\uparrow_x$ = 75% $\uparrow_x$

$\xrightarrow[\downarrow_z \text{ selector}]{}$ 37.5% $\downarrow_z$: $\frac{3}{8}$ = 37.5% are let through.

$\boxed{S2}$ (a) $\left|\frac{1}{\sqrt{2}}(1,1)\frac{1}{2}\begin{pmatrix}1-i\\ \sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{8}\left|1-i+\sqrt{2}\right| = \frac{(1+\sqrt{2})^2+1}{8} = \frac{1}{2}+\frac{\sqrt{2}}{4}.$

(b) $\left|\frac{1}{\sqrt{2}}(1,-i)\frac{1}{2}\begin{pmatrix}1-i\\ \sqrt{2}\end{pmatrix}\right|^2 = \frac{1}{8}\left|1-i-i\sqrt{2}\right| = \frac{1+(1+\sqrt{2})^2}{8} = \frac{1}{2}+\frac{\sqrt{2}}{4}.$

(c) $\left|(1,0)\frac{1}{2}\begin{pmatrix}1-i\\ \sqrt{2}\end{pmatrix}\right|^2 = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2}.$

$\boxed{S3}$ $f(\sigma_x) = a + b\sigma_x$, $f(-\sigma_x) = a - b\sigma_x$ | $f(\sigma_x)\sigma_z = a\sigma_z + b\sigma_x\sigma_z = a\sigma_z - b\sigma_z\sigma_x$
$= \sigma_z f(-\sigma_x).$

$\boxed{S4}$ $\langle x|X = x\langle x|$, $\langle x|P = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|$: $X \to x$, $P \to \frac{\hbar}{i}\frac{\partial}{\partial x}$,
$XP + PX = x\frac{\hbar}{i}\frac{\partial}{\partial x} + \frac{\hbar}{i}\frac{\partial}{\partial x}x = \frac{\hbar}{i}\left(1 + 2x\frac{\partial}{\partial x}\right).$

$\boxed{S5}$

$\mathcal{H}\psi = \begin{pmatrix}0\\ -2E/3\\ 0\end{pmatrix}$, $\langle H\rangle = \psi^\dagger\mathcal{H}\psi = -\frac{2}{9}E$,

$\langle H^2\rangle = (\mathcal{H}\psi)^\dagger(\mathcal{H}\psi) = \frac{4}{9}E^2.$

$\therefore \delta H = \sqrt{\frac{4}{9}E^2 - \frac{4}{81}E^2} = \frac{4}{9}\sqrt{2}\,E.$

Question EXAM/2

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$\boxed{L1}$ (a) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-i\end{pmatrix}$, $\frac{1}{\sqrt{2}}(1,0,i)$ to eigenvalue $3\hbar\omega$,

$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\i\end{pmatrix}$, $\frac{1}{\sqrt{2}}(1,0,-i)$ to eigenvalue $\hbar\omega$,

$\begin{pmatrix}0\\1\\0\end{pmatrix}$, $(0,1,0)$ to eigenvalue $2\hbar\omega$.

(b) the eigenvalues $\hbar\omega$, $2\hbar\omega$, $3\hbar\omega$.

(c) $\text{prob}(\hbar\omega) = \left|\frac{1}{\sqrt{2}}(1,0,-i)\psi_0\right|^2 = 0$,

$\text{prob}(2\hbar\omega) = \left|(0,1,0)\psi_0\right|^2 = \frac{1}{9}$,

$\text{prob}(3\hbar\omega) = \left|\frac{1}{\sqrt{2}}(1,0,i)\psi_0\right|^2 = \frac{8}{9}$.

(d) $$\psi_0 = \underbrace{\frac{1}{3}\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{eigenvalue } 2\hbar\omega} + \underbrace{\frac{i\sqrt{8}}{3}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-i\end{pmatrix}}_{\text{eigenvalue } 3\hbar\omega}$$

$$\therefore \psi(t) = e^{-2i\omega t}\frac{1}{3}\begin{pmatrix}0\\1\\0\end{pmatrix} + e^{-3i\omega t}\frac{2i}{3}\begin{pmatrix}1\\0\\-i\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}2i\,e^{-3i\omega t}\\ e^{-2i\omega t}\\ 2\,e^{-3i\omega t}\end{pmatrix}.$$

(e) $$p(t) = \left|\frac{1}{9}\left(4e^{-3i\omega t} + e^{-2i\omega t} + 4e^{-3i\omega t}\right)\right|^2$$

$$= \frac{1}{81}\left|8e^{-i\omega t/2} + e^{i\omega t/2}\right|^2 = \frac{65+16\cos(\omega t)}{81}.$$

Question EXAM/3

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$\boxed{L2}$ (a) $1 = \int dx\, |\psi(x)|^2 = C^2 \int dx\, e^{-\frac{1}{2}(x/a)^2} = C^2 \sqrt{2\pi}\, a$,

$$C = \frac{(2\pi)^{-1/4}}{\sqrt{a}}.$$

(b) $$\psi(p) = \int dx \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} C e^{-\frac{1}{4}(x/a)^2 + ikx}$$

$$= \frac{C}{\sqrt{2\pi\hbar}} \int dx\, e^{-\frac{1}{4}\left(\frac{x}{a} + 2ipa/\hbar - 2ika\right)^2} \cdot e^{-(p/\hbar - k)^2 a^2}$$

$$= \frac{C}{\sqrt{2\pi\hbar}} \sqrt{\pi}\, 2a\, e^{-(p/\hbar - k)^2 a^2}$$

$$= \frac{(2/\pi)^{1/4}}{\sqrt{\hbar/a}} e^{-(p/\hbar - k)^2 a^2}.$$

(c) $\langle X \rangle = 0$, $\langle P \rangle = \frac{(2/\pi)^{-\frac{1}{2}}}{\hbar/a} \int dp\, \underbrace{p}_{= p - \hbar k + \hbar k} e^{-2(p/\hbar - k)^2 a^2}$

$$= \frac{(2/\pi)^{1/2}}{\hbar/a} \hbar k \sqrt{\frac{\pi}{2(a/\hbar)^2}} = \hbar k.$$

(d) $(\delta X)^2 = \langle X^2 \rangle = C^2 \int dx\, x^2 e^{-\frac{1}{2}(x/a)^2}$

$$= C^2 \frac{1}{2} \sqrt{\pi (2a^2)^3} = C^2 \sqrt{2\pi}\, a^3 = a^2; \ \delta X = a$$

$$(\delta P)^2 = \langle (P - \hbar k)^2 \rangle = \frac{(2/\pi)^{1/2}}{\hbar/a} \int dp\, p^2 e^{-2p^2 (a/\hbar)^2}$$

$$= \frac{\sqrt{2/\pi}}{\hbar/a} \frac{1}{2} \sqrt{\frac{\pi}{(2a^2/\hbar^2)^3}} = \frac{1}{4}\frac{\hbar^2}{a^2}; \ \delta P = \frac{\hbar}{2a}.$$

(e) $\langle (XP + PX) \rangle = 2\,\mathrm{Re} \langle XP \rangle = 2\,\mathrm{Re} \int dx\, \psi(x)^* x \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$

$$= i\hbar \int dx\, x \left(\frac{\partial \psi^*}{\partial x} \psi - \psi^* \frac{\partial \psi}{\partial x} \right) = i\hbar \int dx\, x\, \psi^2 \underbrace{\frac{\partial}{\partial x} \frac{\psi^*}{\psi}}_{= \frac{\partial}{\partial x} e^{-2ikx} = -2ik \frac{\psi^*}{\psi}}$$

$$= 2\hbar k \int dx\, x\, |\psi(x)|^2 = 2 \langle P \rangle \langle X \rangle.$$

They are the same. (And both are zero.)

Question EXAM/4

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$\boxed{3}$(a) Restricted by $\delta X \, \delta P \geq \hbar/2$ (Heisenberg)

(b) $\frac{d}{dt} P(t) = F$, $P(t) = P(0) + Ft$,

$\frac{d}{dt} X(t) = \frac{1}{M} P(t)$, $X(t) = X(0) + \frac{t}{M} P(0) + \frac{F}{2M} t^2$.

(c) $\langle X(t) \rangle = \langle X(0) \rangle + \frac{t}{M} \langle P(0) \rangle + \frac{F}{2M} t^2 = \frac{Ft^2}{2M}$

$$\langle X(t)^2 \rangle = \langle X(0)^2 \rangle + \frac{t^2}{M^2} \langle P(0)^2 \rangle + \left(\frac{Ft^2}{2M}\right)^2$$
$$+ \frac{t}{M} \langle (X(0) P(0) + P(0) X(0)) \rangle$$
$$+ \frac{Ft^3}{M^2} \langle P(0) \rangle + \frac{Ft^2}{M} \langle X(0) \rangle$$
$$= (\delta X)^2 + \left(\frac{t}{M} \delta P\right)^2 + \langle X(t) \rangle^2$$

$$\therefore \; \delta X(t) = \sqrt{(\delta X)^2 + \left(\frac{t}{M} \delta P\right)^2}$$

$\langle P(t) \rangle = \langle P(0) \rangle + Ft = Ft$

$$\langle P(t)^2 \rangle = \langle P(0)^2 \rangle + 2Ft \langle P(0) \rangle + (Ft)^2$$
$$= (\delta P)^2 + \langle P(0)^2 \rangle$$

$$\therefore \; \delta P(t) = \delta P$$

Brief comment: $\delta X(t)$ and $\delta P(t)$ do not involve F; therefore, the spreading of the wave function is exactly the same as that for force-free motion.