Problem 1 (15 points)
Kets $|a\rangle$ and $|b\rangle$ are orthonormal. We represent superposition kets by columns in accordance with

$$|\ \rangle = |a\rangle\psi_a + |b\rangle\psi_b = (|a\rangle, |b\rangle)\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} \mathrel{\widehat{=}} \begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}$$

and statistical operators by 2×2 matrices in accordance with

$$\rho = (|a\rangle, |b\rangle)\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}\begin{pmatrix}\langle a|\\ \langle b|\end{pmatrix} \mathrel{\widehat{=}} \begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix} \equiv S.$$

Find the matrices S_1, S_2, and S_3 of the statistical operators for the mixtures blended from

(1) $\frac{3}{7}$ of $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ and $\frac{4}{7}$ of $\begin{pmatrix}1\\ 0\end{pmatrix}$;

(2) $\frac{1}{7}$ of $\frac{1}{\sqrt{10}}\begin{pmatrix}1\\ -3\end{pmatrix}$ and $\frac{6}{7}$ of $\frac{1}{\sqrt{10}}\begin{pmatrix}3\\ 1\end{pmatrix}$;

(3) $\frac{12}{77}$ of $\begin{pmatrix}0\\ 1\end{pmatrix}$ and $\frac{65}{77}$ of $\frac{1}{\sqrt{130}}\begin{pmatrix}11\\ 3\end{pmatrix}$.

Do you get one, two, or three different mixtures?

Problem 2 (15 points)
Operator A is such that $A^2 = A$.

(a) What are the eigenvalues of A?
(b) Explain why all functions of A are of the linear form $f(A) = f_0 + f_1 A$ with numerical coefficients f_0 and f_1.
(c) Determine f_0 and f_1 for $f(A) = \cos(\pi A)$.

Problem 3 (20 points)
Show that

$$\mathrm{e}^{\mathrm{i}\lambda P^3/\hbar}\mathrm{e}^{-\mu X}\mathrm{e}^{-\mathrm{i}\lambda P^3/\hbar} = \mathrm{e}^{-\mu X - 3\mu\lambda P^2}$$

where λ and μ are numerical constants.

Problem 4 (25 points)
The state of a system is specified by its momentum wave function $\psi(p) = \langle p|\ \rangle$.

(a) Express the expectation values $\langle X\rangle$, $\langle X^2\rangle$, and $\langle PX\rangle$ by integrals involving $\psi(p)$ and $\partial\psi(p)/\partial p$.
(b) Evaluate these integrals for $\psi(p) = \begin{cases} 2\sqrt{a^3}\,p\,\mathrm{e}^{-ap} & \text{for } p > 0\,,\\ 0 & \text{for } p < 0\,,\end{cases}$
where $a > 0$ is a constant parameter.

[Hint: You may have a use for $\int_0^\infty \mathrm{d}x\, x^n \mathrm{e}^{-x} = n!$ for $n > -1$.]

Problem 5 (25 points)
Operator R is such that $\langle x|R = 2\langle -x|$ for all position bras $\langle x|$.

(a) Determine the $X;P$-ordered form of R.
(b) Calculate the trace of R with the aid of a suitable phase-space integration.