Question *Test 2/1*

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$\boxed{1}$ These are Gaussian wave functions so that they are minimum uncertainty wave functions. Therefore, the position wave function for ket $|2\rangle$ is immediately available: It has the appearance of $\langle x|1\rangle$ but with a replaced by $\frac{\hbar}{2b}$,

$$\langle x|2\rangle = \frac{(2\pi)^{-1/4}}{\sqrt{\frac{\hbar}{2b}}} e^{-\left(\frac{bx}{\hbar}\right)^2} .$$

Alternatively, you can get this by Fourier transforming $\langle p|2\rangle$.

Then

$$\langle 1|2\rangle = \int dx \langle 1|x\rangle\langle x|2\rangle$$

$$= \frac{(2\pi)^{-1/2}}{\sqrt{\frac{\hbar a}{2b}}} \int dx\, e^{-\left(\frac{x}{2a}\right)^2 - \left(\frac{bx}{\hbar}\right)^2}$$

$$= \sqrt{\frac{2b}{2\pi\hbar a}} \sqrt{\frac{\pi}{\left(\frac{1}{2a}\right)^2 + \left(\frac{b}{\hbar}\right)^2}} = \sqrt{\frac{b}{\hbar a} \frac{(2\hbar a)^2}{\hbar^2 + (2ab)^2}}$$

$$= \sqrt{\frac{4\hbar ab}{\hbar^2 + (2ab)^2}}$$

and $$|\langle 1|2\rangle|^2 = \frac{4ab/\hbar}{1 + (2ab/\hbar)^2} .$$

For $a = \frac{\hbar}{2b}$ or $2ab/\hbar = 1$, we have $|1\rangle = |2\rangle$, and get $|\langle 1|2\rangle|^2 = 1$, as we should.

$\boxed{2}$ (a) $$\frac{d}{dt} P = -\frac{\partial H}{\partial X} = 2\Omega P \rightsquigarrow P(t) = P(t_0) e^{2\Omega T},$$

$$\frac{d}{dt} X = \frac{\partial H}{\partial P} = -2\Omega X \rightsquigarrow X(t) = X(t_0) e^{-2\Omega T}.$$

(b) $$[X(t), P(t_0)] = e^{-2\Omega T} [X(t_0), P(t_0)] = i\hbar e^{-2\Omega T}.$$

Question Test 2/2

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(c) $i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \langle x, t | H | p, t_0 \rangle$

with H at time t:

$$H = -\Omega\left(X(t)P(t) + P(t)X(t)\right)$$
$$= -2\Omega X(t)P(t) + i\hbar\Omega$$
$$= -2\Omega X(t)e^{2\Omega T}P(t_0) + i\hbar\Omega$$

so that

$$i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \left(-2\Omega x p e^{2\Omega T} + i\hbar\Omega\right)\langle x, t | p, t_0 \rangle$$

or

$$\frac{\partial}{\partial t} \log \langle x, t | p, t_0 \rangle = i \frac{xp}{\hbar} 2\Omega e^{2\Omega T} + \Omega$$
$$= \frac{\partial}{\partial t}\left(i \frac{xp}{\hbar} e^{2\Omega T} + \Omega T\right).$$

It follows that

$$\langle x, t | p, t_0 \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{xp}{\hbar}e^{2\Omega T}} e^{\Omega T}$$

after incorporating the initial condition

$$\langle x, t | p, t_0 \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar} \text{ for } T \to 0.$$

Now,

$$\langle x, t | x', t_0 \rangle = \int dp \langle x, t | p, t_0 \rangle \langle p, t_0 | x', t_0 \rangle$$
$$= \int dp \frac{1}{\sqrt{2\pi\hbar}} e^{ixpe^{2\Omega T}/\hbar} e^{\Omega T} \frac{e^{-ipx'/\hbar}}{\sqrt{2\pi\hbar}}$$
$$= e^{\Omega T} \delta\left(x e^{2\Omega T} - x'\right)$$
$$= \delta\left(x e^{\Omega T} - x' e^{-\Omega T}\right).$$

Question Test 2/3

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3 We have $\rho(A^+(t), A(t), t) = \rho(A^+(t_0), A(t_0), t_0)$

$$= e^{-(A^+(t_0)-\alpha^*);(A(t_0)-\alpha)}$$

and $A^+(t) = e^{i\omega T}A^+(t_0)$, $A(t) = e^{-i\omega T}A(t_0)$

or $A^+(t_0) = e^{-i\omega T}A^+(t)$, $A(t_0) = e^{i\omega T}A(t)$,

so that

$$\rho(A^+, A, t) = e^{-(A^+e^{-i\omega T}-\alpha^*);(Ae^{i\omega T}-\alpha)}$$

$$= e^{-(A^+-\alpha^*e^{i\omega T});(A-\alpha e^{-i\omega T})}.$$

4 (a) With

$$A|\rangle = |0\rangle\frac{1}{\sqrt{2}}, \quad \langle|A^+ = \frac{1}{\sqrt{2}}\langle 0|$$

$$A^2|\rangle = 0, \quad \langle|A^{+2} = 0,$$

we get

$$\langle A\rangle = \langle|0\rangle\frac{1}{\sqrt{2}} = \frac{1}{2}, \quad \langle A^+\rangle = \frac{1}{2}$$

$$\langle A^2\rangle = 0, \quad \langle A^{+2}\rangle = 0$$

(b) Also will need $\langle A^+A\rangle = \frac{1}{2}$, $\langle AA^+\rangle = \frac{3}{2}$

in

$$\langle X^2\rangle = \langle \frac{\ell^2}{2}(A^{+2}+A^+A+AA^++A^2)\rangle = \ell^2,$$

$$\langle P^2\rangle = \langle \frac{\hbar^2}{2\ell^2}(-A^{+2}+A^+A+AA^+-A^2)\rangle = \left(\frac{\hbar}{\ell}\right)^2.$$

With $\langle X\rangle = \frac{\ell}{\sqrt{2}}\langle(A^++A)\rangle = \frac{\ell}{\sqrt{2}}$

and $\langle P\rangle = \frac{\hbar}{\sqrt{2}\ell}\langle(iA^+-iA)\rangle = 0$

this gives

$$\left.\begin{aligned} \delta X &= \sqrt{\ell^2-\left(\frac{\ell}{\sqrt{2}}\right)^2} = \frac{\ell}{\sqrt{2}} \\ \delta P &= \sqrt{\left(\frac{\hbar}{\ell}\right)^2-0^2} = \frac{\hbar}{\ell} \end{aligned}\right\} \delta X\delta P = \frac{\hbar}{\sqrt{2}} > \frac{\hbar}{2}.$$

Question Test 2/4

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5 (a) Use the completeness relation for coherent states,

$$\mathrm{tr}\{F\} = \mathrm{tr}\left\{F \int \frac{dx\,dp}{2\pi\hbar} \frac{|a'\rangle\langle a^*|}{\langle a^*|a'\rangle}\Big|_{\text{chosen parametrization}} \right\}$$

$$= \int \frac{dx\,dp}{2\pi\hbar} \frac{\langle a'|F|a^*\rangle}{\langle a^*|a'\rangle}$$

$$= \int \frac{dx\,dp}{2\pi\hbar} f(a^*, a')$$

and take the parametrization $a' = \frac{x}{\ell}$, $a^* = -i\frac{\ell p}{\hbar}$ to get

$$\mathrm{tr}\{F\} = \int \frac{dx\,dp}{2\pi\hbar} f\left(-i\frac{\ell p}{\hbar}, \frac{x}{\ell}\right).$$

(b)

$$\mathrm{tr}\{e^{-\lambda A^\dagger; A}\} = \int \frac{dx\,dp}{2\pi\hbar} e^{-\lambda\left(-\frac{i\ell p}{\hbar}\right)\frac{x}{\ell}}$$

$$= \int \frac{dx\,dp}{2\pi\hbar} e^{i\lambda xp/\hbar} = \int dx\, \delta(\lambda x) = \frac{1}{\lambda}.$$