Question *Exam/1*

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$\boxed{1}$ (a) $1 = \mathrm{tr}\{\rho\} = Z(\beta) \sum_{n=0}^{\infty} e^{-n\beta} = \dfrac{Z(\beta)}{1-e^{-\beta}}$

gives $Z(\beta) = 1 - e^{-\beta}$.

(b) $\langle (A^{+}A)^{k} \rangle = Z(\beta)\, \mathrm{tr}\{ e^{-\beta A^{+}A} (A^{+}A)^{k} \}$

$= Z(\beta)\, \mathrm{tr}\{ \underbrace{e^{-\beta A^{+}A} A^{+}}_{= A^{+} e^{-\beta(A^{+}A+1)}} A (A^{+}A)^{k-1} \}$

$= Z(\beta)\, \mathrm{tr}\{ A^{+} e^{-\beta(A^{+}A+1)} (AA^{+})^{k-1} A \}$

move over, exploiting the cyclic property of the trace

$= Z(\beta)\, e^{-\beta}\, \mathrm{tr}\{ e^{-\beta A^{+}A} (AA^{+})^{k} \}$

$= e^{-\beta} \langle (AA^{+})^{k} \rangle = e^{-\beta} \langle (A^{+}A+1)^{k} \rangle.$

(c) $k=1$: $\langle A^{+}A \rangle = e^{-\beta} (\langle A^{+}A \rangle + 1)$,

so that $\langle A^{+}A \rangle = \dfrac{e^{-\beta}}{Z(\beta)} = \dfrac{1}{e^{\beta}-1}$;

$k=2$: $\langle (A^{+}A)^{2} \rangle = e^{-\beta} (\langle (A^{+}A)^{2} \rangle + 2 \langle A^{+}A \rangle + 1)$

so that $\langle (A^{+}A)^{2} \rangle = \dfrac{e^{-\beta}}{Z(\beta)} (2 \langle A^{+}A \rangle + 1)$

$= \langle A^{+}A \rangle (2 \langle A^{+}A \rangle + 1)$

and

$\delta(A^{+}A) = \sqrt{ \langle (A^{+}A)^{2} \rangle - \langle A^{+}A \rangle^{2} } = \sqrt{ \langle A^{+}A \rangle (\langle A^{+}A \rangle + 1) }$

$= \dfrac{e^{\beta/2}}{e^{\beta}-1} = \dfrac{1}{2 \sinh(\beta/2)}.$

Question Exam/2

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$\boxed{2}$ (a) $\langle l,m|\frac{1}{\hbar}(L_1 \pm iL_2)|l,m\rangle = \langle l,m|l,m\pm1\rangle\sqrt{(l\mp m)(l\pm m+1)}$
$= 0$

so that $\langle L_1\rangle = 0$, $\langle L_2\rangle = 0$.
Likewise $\langle (L_1 \pm iL_2)^2\rangle = 0$, so that
$\langle (L_1+iL_2)^2\rangle + \langle (L_1-iL_2)^2\rangle = 2(\langle L_1^2\rangle - \langle L_2^2\rangle) = 0$,
implying $\langle L_1^2\rangle = \langle L_2^2\rangle$.
Further, $L_1^2 + L_2^2 = \vec{L}^2 - L_3^2$, so that
$\langle L_1^2\rangle + \langle L_2^2\rangle = \hbar^2(l(l+1) - m^2)$, and it follows that

$$\delta L_1 = \delta L_2 = \frac{\hbar}{\sqrt{2}}\sqrt{l(l+1)-m^2} .$$

Largest value for $m=0$: $\hbar\sqrt{l(l+1)/2}$;
smallest value for $m=\pm l$: $\hbar\sqrt{l/2}$.

(b) $L_+|\,\rangle = |1,0\rangle\hbar$, $L_-|\,\rangle = |1,0\rangle\hbar$,
$L_+^2|\,\rangle = |1,1\rangle\sqrt{2}\hbar^2$, $L_-^2|\,\rangle = |1,-1\rangle\sqrt{2}\hbar^2$,
so that $\langle L_+\rangle = 0$, implying $\langle L_1\rangle = 0, \langle L_2\rangle = 0$,
and $\langle L_+^2\rangle = \hbar^2$, $\langle L_-^2\rangle = \hbar^2$, so that
$\langle L_1^2\rangle - \langle L_2^2\rangle = \frac{1}{2}(\langle L_+^2\rangle + \langle L_-^2\rangle) = \hbar^2$.
Further, $\langle L_3\rangle = \frac{1}{2}(+\hbar) + \frac{1}{2}(-\hbar) = 0$,
$\langle L_3^2\rangle = \frac{1}{2}\hbar^2 + \frac{1}{2}(-\hbar)^2 = \hbar^2$,
$\langle \vec{L}^2\rangle = \langle L_1^2\rangle + \langle L_2^2\rangle + \langle L_3^2\rangle = 2\hbar^2$,

so that $\langle L_1^2\rangle = \hbar^2$, $\langle L_2^2\rangle = 0$,
and the respective spreads are

$$\delta L_1 = \hbar,\ \delta L_2 = 0,\ \delta L_3 = \hbar .$$

Conclusion: The state is eigenstate of L_2 with eigenvalue $0\hbar$.

Question Exam/3

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[3] (a) For such Gaussian wave functions we know that $\langle P \rangle = 0$ and $\langle P^2 \rangle = (\frac{\hbar}{2a})^2$, so that

$$E_{kin} = \frac{\hbar^2}{8Ma^2} = \frac{(\hbar k)^2}{2M}\frac{1}{4(ka)^2} = \frac{(\hbar k)^2}{2M}\frac{1}{4y}$$

is immediate. Further,

$$\langle E_{pot} \rangle = -\frac{(\hbar k)^2}{\sqrt{2}\,M}\frac{1}{\sqrt{2\pi}}\frac{1}{a}\underbrace{\int dx\, e^{-\left(\frac{k^2}{2}+\frac{1}{2a^2}\right)x^2}}_{=\sqrt{\frac{2\pi}{k^2+1/a^2}}}$$

$$= -\frac{(\hbar k)^2}{\sqrt{2}\,M}\frac{1}{\sqrt{1+(ka)^2}} = -\frac{(\hbar k)^2}{2M}\sqrt{\frac{2}{1+y}}\,.$$

(b) Rayleigh-Ritz: $E_{kin} + E_{pot} \geq E_0$, so that here

$$E_0 \leq \frac{(\hbar k)^2}{2M}\left(\frac{1}{4y} - \sqrt{\frac{2}{1+y}}\right).$$

The right-hand side is smallest if y is such that $-\frac{1}{4y^2} + \sqrt{\frac{1}{2(1+y)^3}} = 0$, or $\underline{y=1}$.

Thus, the lowest upper bound of this kind is

$$E_0 \leq \frac{(\hbar k)^2}{2M}\left(\frac{1}{4} - 1\right) = -\frac{3(\hbar k)^2}{8M}\,.$$

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[4] The unperturbed states are the eigenstates of $\hbar\omega A^\dagger A$, so they are the usual Fock states of the harmonic oscillator: $|n^{(0)}\rangle = \frac{1}{\sqrt{n!}} A^{\dagger n} |0^{(0)}\rangle$, and we have

$$\langle m^{(0)} | H_1 | n^{(0)} \rangle = \frac{i\hbar}{2}\Omega \langle m^{(0)} | (A^{\dagger 2} - A^2) | n^{(0)} \rangle$$

$$= \frac{i\hbar}{2}\Omega \Big[\delta_{m,n+2}\sqrt{m(n+1)} - \delta_{m,n-2}\sqrt{n(m+1)} \Big].$$

Question Exam/4

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(a) $E_n = E_n^{(0)} + E_n^{(1)} + \underbrace{E_n^{(2)} + \cdots}_{\text{neglected}}$

$$= n\hbar\omega + \underbrace{\langle n^{(0)}|H_1|n^{(0)}\rangle}_{0,\text{ here}} - \sum_{m(\neq n)} \frac{|\langle m^{(0)}|H_1|n^{(0)}\rangle|^2}{\hbar\omega m - \hbar\omega n}$$

$$= n\hbar\omega - \frac{\hbar^2\Omega^2}{4}\frac{1}{\hbar\omega}\left(\underbrace{\frac{(n+1)(n+2)}{(n+2)-n}}_{\text{for } m=n+2} + \underbrace{\frac{n(n-1)}{(n-2)-n}}_{\text{for } m=n-2}\right)$$

$$= n\hbar\omega - \frac{\hbar\Omega^2}{4\omega}(2n+1) = \hbar\omega\left[n - \left(\frac{\Omega}{\omega}\right)^2\left(\frac{n}{2}+\frac{1}{4}\right)\right].$$

(b) Here $E_n = n\hbar\omega - \frac{\hbar^2\Omega^2}{4}\left(\frac{(n+1)(n+2)}{(n+2)\hbar\omega - E_n} + \frac{n(n-1)}{(n-2)\hbar\omega - E_n}\right)$

for $n=0$: $E_0 = -\frac{\hbar^2\Omega^2}{2}\frac{1}{2\hbar\omega - E_0}$

or $E_0(E_0 - 2\hbar\omega) = (E_0 - \hbar\omega)^2 - (\hbar\omega)^2 = \frac{1}{2}\hbar^2\Omega^2$

so that $E_0 = \hbar\omega - \sqrt{(\hbar\omega)^2 + \frac{1}{2}(\hbar\Omega)^2}$

$$= \hbar\omega\left[1 - \sqrt{1 + \tfrac{1}{2}(\Omega/\omega)^2}\right]$$

$$= \hbar\omega\left[-\frac{1}{4}\left(\frac{\Omega}{\omega}\right)^2 + \frac{1}{32}\left(\frac{\Omega}{\omega}\right)^4 + \cdots\right].$$

(c) $E_n = \hbar\omega\left[(n+\tfrac{1}{2})\sqrt{1-(\Omega/\omega)^2} - \tfrac{1}{2}\right]$

$$= \hbar\omega\left[n - (n+\tfrac{1}{2})\tfrac{1}{2}\left(\frac{\Omega}{\omega}\right)^2 - (n+\tfrac{1}{2})\tfrac{1}{8}\left(\frac{\Omega}{\omega}\right)^4 + \cdots\right].$$

So, the $(\Omega/\omega)^2$ terms of (a) and (b) are correct, as they should be.

Brillouin-Wigner does not do very well here for the higher-order terms, as the sign of the $(\Omega/\omega)^4$ term is wrong.