Question. Test 2/1

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$\boxed{1}$ (a) In the ground state $|0\rangle$ of H_0 we have

$$\langle H_1 \rangle = \hbar\Omega \underbrace{\langle 0|A^{+}|0\rangle}_{=0} + \hbar\Omega^{*} \langle 0|\underbrace{A|0\rangle}_{=0}$$

$$= 0,$$

$$\langle H_1^2 \rangle = (\hbar\Omega)^2 \underbrace{\langle 0|A^{+2}|0\rangle}_{=0} + (\hbar\Omega^{*})^2 \langle 0|\underbrace{A^2|0\rangle}_{=0}$$

$$+ (\hbar|\Omega|)^2 \underbrace{\langle 0|A^{+}}_{=0}\underbrace{A|0\rangle}_{=0} + (\hbar|\Omega|)^2 \underbrace{\langle 0|AA^{+}|0\rangle}_{=1}$$

$$= (\hbar|\Omega|)^2,$$

so that $\delta H = \delta H_1 = \hbar|\Omega|$

(b) To first order in Ω we have

$$S(T) = 1 - \frac{i}{\hbar}\int_0^T dt\, e^{iH_0 t/\hbar} H_1 e^{-iH_0 t/\hbar}$$

$$= 1 - i\int_0^T dt\, e^{i\omega t A^{+}A}(\Omega A^{+} + \Omega^{*}A)e^{-i\omega t A^{+}A}$$

$$= 1 - i\int_0^T dt\,(\Omega e^{i\omega t}A^{+} + \Omega^{*}e^{-i\omega t}A)$$

$$= 1 - i\Omega \underbrace{\frac{e^{i\omega T}-1}{i\omega}}_{= e^{i\omega T/2}\frac{\sin(\omega T/2)}{\omega/2}} A^{+} - i\Omega^{*}\frac{e^{-i\omega T}-1}{-i\omega}A$$

$$= 1 - \frac{2i}{\omega}\sin\left(\frac{\omega T}{2}\right)\left(\Omega e^{i\omega T/2}A^{+} + \Omega^{*}e^{-i\omega T/2}A\right).$$

(c) Probability for nth excited state of H_0 after elapse of time T is

$$p(n,T) = |\langle n|e^{-iHT/\hbar}|0\rangle|^2$$

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where $\langle n|e^{-iHT/\hbar}|0\rangle = \langle n|e^{-iH_0T/\hbar} S(T)|0\rangle$

$= e^{-in\omega T} \langle n|S(T)|0\rangle$

with

$$\langle n|S(T)|0\rangle = \underbrace{\langle n|0\rangle}_{\delta_{n,0}} - \frac{2i}{\omega}\sin\left(\frac{\omega T}{2}\right)\Omega e^{i\omega T/2}\underbrace{\langle n|A^{+}|0\rangle}_{\delta_{n,1}} - \frac{2i}{\omega}\sin\left(\frac{\omega T}{2}\right)\Omega^{*} e^{-i\omega T/2}\underbrace{\langle n|A|0\rangle}_{=0}$$

$$= \delta_{n,0} - 2i\frac{\Omega}{\omega}\sin\left(\frac{\omega T}{2}\right) e^{i\omega T/2}\delta_{n,1}$$

$$= \begin{cases} -2i\frac{\Omega}{\omega}\sin\left(\frac{\omega T}{2}\right)e^{i\omega T/2} & \text{for } n=1 \\ 0 & \text{for } n>1 \end{cases}$$

to first order in Ω. Therefore,

$$p(1,T) = \left(\frac{2\Omega}{\omega}\sin\left(\frac{\omega T}{2}\right)\right)^2$$

$$p(2,T) = p(3,T) = p(4,T) = \cdots = 0.$$

$\boxed{2}$ (a) We have $|\psi(\vec{r},t)|^2 = C^2 \frac{x^2+y^2}{r^2} e^{-2r/a}$

$$= C^2 (\sin\vartheta)^2 e^{-2r/a}$$

in spherical coordinates, so that

$$1 = \int (d\vec{r})\, |\psi|^2 = C^2\, 2\pi \underbrace{\int_0^{\pi} d\vartheta\, (\sin\vartheta)^3}_{4/3} \underbrace{\int_0^{\infty} dr\, r^2 e^{-2r/a}}_{2!\left(\frac{a}{2}\right)^3}$$

$$= C^2 \frac{2\pi}{3} a^3$$

giving $C = \sqrt{\frac{3}{2\pi a^3}}$.

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(b) $\rho(\vec{r},t) = |\psi(\vec{r},t)|^2 = \frac{3}{2\pi a^3} \frac{x^2+y^2}{r^2} e^{-2r/a}$;

$\vec{j} = \frac{\hbar}{M} \operatorname{Im}\left(\psi(\vec{r},t)^* \vec{\nabla} \psi(\vec{r},t)\right)$

has no contribution from the r dependence because $\psi^* \frac{\partial}{\partial r} \psi$ is real; so we only need to consider the contributions that come from differentiating the factor $x+iy$:

$$\vec{j} = \frac{\hbar}{M} \frac{C^2}{r^2} e^{-2r/a} \operatorname{Im}\left((x-iy) \underbrace{\vec{\nabla}(x+iy)}_{= \vec{e}_x + i\vec{e}_y}\right)$$

$$= \frac{\hbar}{M} \frac{3}{2\pi a^3} \frac{e^{-2r/a}}{r^2} (x\vec{e}_y - y\vec{e}_x)$$

where $\vec{e}_x, \vec{e}_y$ are the unit vectors for the x and y directions.

(c) Since $\frac{\partial}{\partial t}\rho = 0$ here, we need to check that $\vec{\nabla}\cdot\vec{j} = 0$:

$$\vec{\nabla}\cdot\vec{j} = \frac{\hbar}{M} \frac{3}{2\pi a^3} \left[\frac{\partial}{\partial x}\left(-y \frac{e^{-2r/a}}{r^2}\right) + \frac{\partial}{\partial y}\left(x \frac{e^{-2r/a}}{r^2}\right)\right],$$

where we now use $\frac{\partial}{\partial x} f(r) = \frac{x}{r} \frac{\partial}{\partial r} f(r)$

and $\frac{\partial}{\partial y} f(r) = \frac{y}{r} \frac{\partial}{\partial r} f(r)$

to arrive at

$$\vec{\nabla}\cdot\vec{j} = \frac{\hbar}{M} \frac{3}{2\pi a^3} \left(-y\frac{x}{r} + x\frac{y}{r}\right) \frac{\partial}{\partial r} \frac{e^{-2r/a}}{r^2}$$

$$= 0 \text{, indeed.}$$

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(d) General argument

$$\frac{d}{dt}\int (d\vec{r})\,\vec{r}\,\rho(\vec{r},t) = \int (d\vec{r})\,\vec{r}\,\frac{\partial}{\partial t}\rho(\vec{r},t)$$

$$= -\int (d\vec{r})\,\vec{r}\,\vec{\nabla}\cdot\vec{j}(\vec{r},t)$$

$$= -\int (d\vec{r})\Big[\underbrace{\vec{\nabla}\cdot\big(\vec{j}(\vec{r},t)\,\vec{r}\big)}_{\to\, 0 \text{ since it is equal to a surface integral at infinity}} - \underbrace{\vec{j}(\vec{r},t)\cdot\vec{\nabla}\vec{r}}_{=\vec{j}(\vec{r},t)}\Big]$$

$$= \int (d\vec{r})\,\vec{j}(\vec{r},t), \text{ done.}$$

Explicit calculation for ρ and $\vec{j}$ of (b)

$\frac{d}{dt}\int (d\vec{r})\,\vec{r}\,\rho(\vec{r},t) = 0$ since $\rho(\vec{r},t)$ does not depend on time here;

$\int (d\vec{r})\,\vec{j}(\vec{r},t) = 0$ since the $\vec{j}$ in (b) is an odd function of $\vec{r}$, $\vec{j}(\vec{r},t) = -\vec{j}(-\vec{r},t)$.

$\boxed{3}$ (a) We have

$$\int (d\vec{r})\,V(\vec{r})\,e^{i\vec{q}\cdot\vec{r}} = V_0\int (d\vec{r})\,e^{-\frac{1}{2}\left(\frac{r}{a}\right)^2 + i\vec{q}\cdot\vec{r}}$$

$$= V_0\int (d\vec{r})\,e^{-\frac{1}{2}\left(\vec{r}/a - ia\vec{q}\right)^2}\,e^{-\frac{1}{2}a^2q^2}$$

$$= V_0\,\sqrt{2\pi a^2}^{\,3}\,e^{-\frac{1}{2}a^2q^2} = \frac{\hbar^2}{2M}(2\pi)^{3/2}a\,e^{-\frac{1}{2}a^2q^2},$$

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so that

$$\frac{d\sigma}{d\Omega} = \left| \frac{M}{2\pi\hbar^2} \int (d\vec{r}\,) V(\vec{r}\,) e^{i\vec{q}\cdot\vec{r}} \right|^2$$

$$= \frac{\pi}{2} a^2 e^{-(aq)^2} .$$

(b)

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int_0^{\pi} d\theta \sin\theta \; \frac{\pi}{2} a^2 e^{-(aq)^2}$$

$$= 2\pi \int_0^{2k} dq \frac{q}{k^2} \frac{\pi}{2} a^2 e^{-(aq)^2}$$

$$= \frac{\pi^2}{2k^2} \int_0^{2k} dq \left(-\frac{\partial}{\partial q} e^{-a^2 q^2} \right)$$

$$= \frac{\pi^2}{2k^2} \left(1 - e^{-4(ka)^2} \right) = \frac{\pi^2}{2} a^2 \frac{1 - e^{-4(ka)^2}}{(ka)^2}$$

where $(ka)^2 = \frac{(\hbar k)^2}{2M} \Big/ \frac{(\hbar/a)^2}{2M} = E/V_0$,

so that

$$\sigma = \frac{\pi^2}{2} a^2 \frac{1 - e^{-4E/V_0}}{E/V_0}$$

(c) $E \ll V_0$: $1 - e^{-4E/V_0} \cong 4 E/V_0$

$$\sigma \cong 2\pi^2 a^2 \quad ;$$

$E \gg V_0$: $\sigma \cong \frac{\pi^2}{2} a^2 \frac{V_0}{E}$.

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$\boxed{4}$ (a) $\begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$ has eigen columns $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$ and $\begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$ to eigenvalues $+1$ and -1, respectively.

So
$$\psi_+(t) = \begin{pmatrix} \cos(\phi(t)) \\ \sin(\phi(t)) \end{pmatrix}$$
$$\psi_-(t) = \begin{pmatrix} -\sin(\phi(t)) \\ \cos(\phi(t)) \end{pmatrix}.$$

(b)
$$i\hbar \frac{\partial}{\partial t}\psi(t) = \mathcal{H}(t)\psi(t) = \hbar\omega\,\alpha(t)\psi_+(t) - \hbar\omega\beta(t)\psi_-(t)$$
$$\stackrel{!}{=} i\hbar\frac{\partial\alpha}{\partial t}\psi_+ + i\hbar\frac{\partial\beta}{\partial t}\psi_- + i\hbar\,\alpha\frac{\partial\psi_+}{\partial t} + i\hbar\,\beta\frac{\partial\psi_-}{\partial t}$$

where $\frac{\partial}{\partial t}\psi_+(t) = \frac{\partial\phi}{\partial t}\psi_-(t) = \frac{\pi}{T}\psi_-(t)$,
$$\frac{\partial}{\partial t}\psi_-(t) = -\frac{\partial\phi}{\partial t}\psi_+(t) = -\frac{\pi}{T}\psi_+(t)$$

so that
$$\omega\alpha\psi_+ - \omega\beta\psi_- = i\left(\frac{\partial\alpha}{\partial t} - \frac{\pi}{T}\beta\right)\psi_+ + i\left(\frac{\partial\beta}{\partial t} + \frac{\pi}{T}\alpha\right)\psi_-$$

implying
$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix} -i\omega\alpha + \frac{\pi}{T}\beta \\ i\omega\beta - \frac{\pi}{T}\alpha \end{pmatrix} = i\begin{pmatrix} -\omega & -i\pi/T \\ i\frac{\pi}{T} & \omega \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

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that is : $\mathcal{M} = \begin{pmatrix} -\omega & -i\pi/T \\ i\pi/T & \omega \end{pmatrix}$.

(c) Since $\mathcal{M}^2 = (\omega^2 + (\pi/T)^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\begin{pmatrix} \alpha(T) \\ \beta(T) \end{pmatrix} = e^{i\mathcal{M}T} \begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix}$$

$$= \left[\cos\left(\sqrt{\omega^2 + (\pi/T)^2}\, T\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \frac{\sin\left(\sqrt{\omega^2 + (\pi/T)^2}\, T\right)}{\sqrt{\omega^2 + (\pi/T)^2}} \mathcal{M} \right] \begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix},$$

to be used for $\psi(0) = \alpha(0)\,\psi_+(0) + \beta(0)\,\psi_-(0)$

$$= \begin{pmatrix} \alpha(0) \\ \beta(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\psi(T) = \alpha(T)\,\psi_+(T) + \beta(T).\psi_-(T)$

$$= \begin{pmatrix} -\alpha(T) \\ -\beta(T) \end{pmatrix},$$

so that

$$\psi(T) = -\cos\left(\sqrt{(\omega T)^2 + \pi^2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \frac{\sin\left(\sqrt{(\omega T)^2 + \pi^2}\right)}{\sqrt{(\omega T)^2 + \pi^2}} \begin{pmatrix} -\omega T \\ i\pi \end{pmatrix} = \begin{pmatrix} \psi_1(T) \\ \psi_2(T) \end{pmatrix},$$

or quite explicitly

$$\psi_1(T) = -\cos\left(\sqrt{(\omega T)^2 + \pi^2}\right) + \frac{i\omega T}{\sqrt{(\omega T)^2 + \pi^2}} \sin\left(\sqrt{(\omega T)^2 + \pi^2}\right),$$

$$\psi_2(T) = \frac{\pi}{\sqrt{(\omega T)^2 + \pi^2}} \sin\left(\sqrt{(\omega T)^2 + \pi^2}\right).$$

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(d) $$|\psi_2(T)|^2 = \frac{\sin\left(\pi\sqrt{1+(\omega T/\pi)^2}\right)^2}{1+(\omega T/\pi)^2};$$

for $\omega T \ll 1$, $\sqrt{1+(\omega T/\pi)^2} \cong 1 + \frac{1}{2}\left(\frac{\omega T}{\pi}\right)^2$

$$\sin\left(\pi\sqrt{\cdot}\right) \cong -\sin\left(\frac{\pi}{2}\left(\frac{\omega T}{\pi}\right)^2\right)$$

$$\cong -\frac{1}{2\pi}(\omega T)^2$$

$$|\psi_2(T)|^2 \cong \frac{(\omega T)^4}{4\pi^2} \quad \text{(very small)};$$

for $\omega T \gg 1$ $\sqrt{1+(\omega T/\pi)^2} \cong \omega T/\pi$

$$|\psi_2(T)|^2 \cong \frac{\sin(\omega T)^2}{(\omega T/\pi)^2} = \left|\frac{\pi}{\omega T}\sin(\omega T)\right|^2$$

$$\propto \left(\frac{1}{\omega T}\right)^2$$

(also quite small).

So we have $|\psi_2(T)|^2 = 0$ in the limit of $T \to 0$ (violent perturbation of very short duration) and $T \to \infty$ (adiabatically slow evolution).