Question Exam/1.

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[1](a) s-wave radial Schrödinger equation

$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r^2} + V(r)\right] u_0(r) = E\, u_0(r)$$

reads here

$$\frac{\partial^2}{\partial r^2} u_0(r) = -k^2 u_0(r) + \frac{1}{a}\delta(r-a)\, u_0(r)$$

so that $\frac{\partial^2}{\partial r^2} u_0(r) = -k^2 u_0(r)$ for $r \neq a$, and

$$u_0(r) = \begin{cases} C \sin(kr) & , \quad 0 < r < a \\ \sin(kr + \delta_0) & , \quad r > a \end{cases}$$

follows immediately. The continuity of $u_0(r)$ at $r = a$ implies

$$\sin(ka + \delta_0) = C \sin(ka)$$

and by integrating over the discontinuity of $\partial u_0 / \partial r$ at $r = a$,

$$\frac{1}{k}\int_{a-0}^{a+0} dr \, \frac{\partial^2}{\partial r^2} u_0(r) = \frac{1}{k}\frac{\partial}{\partial r} u_0(r)\Bigg|_{r=a-0}^{r=a+0}$$

$$= \frac{1}{ka} u_0(a) = \frac{C}{ka}\sin(ka)$$

we get

$$\cos(ka + \delta_0) = C\left[\cos(ka) + \frac{\sin(ka)}{ka}\right].$$

(b) Since $\sigma_0 = \frac{4\pi}{k^2}(\sin\delta_0)^2$, we need to find

$\sin\delta_0 = \sin(ka + \delta_0)\cos(ka) - \cos(ka + \delta_0)\sin(ka)$

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that is

$$\sin\delta_0 = -C\,\frac{1}{ka}\left(\sin(ka)\right)^2 = -C\,ka\left(\frac{\sin(ka)}{ka}\right)^2,$$

where C^2 is available from

$$1 = \sin(ka+\delta_0)^2 + \cos(ka+\delta_0)^2$$

$$= C^2\left[1+\frac{\sin(2ka)}{ka}+\left(\frac{\sin(ka)}{ka}\right)^2\right].$$

Accordingly, $\sigma_0 = \pi a^2 f(ka)$ with

$$f(ka) = \frac{4}{1+\frac{\sin(2ka)}{ka}+\left(\frac{\sin(ka)}{ka}\right)^2}\left(\frac{\sin(ka)}{ka}\right)^4.$$

(c) For $ka \ll 1$, one has $\frac{\sin(ka)}{ka} \cong 1$, $\frac{\sin(2ka)}{ka} \cong 2$ and $f(ka) \cong 1$ follows, so that $\sigma_0 \cong \pi a^2$.

$\boxed{2}$ (a) Since $\vec{J}^2 = \vec{L}^2 + \vec{S}^2 + 2\vec{L}\cdot\vec{S}$, we have

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$$

and since <u>all</u> states in question are eigenstates of $\vec{L}^2$ and $\vec{S}^2 = \frac{3}{4}\hbar^2$, the eigenstates of $\vec{J}^2$ are also eigenstates of $\vec{L}\cdot\vec{S}$, with eigenvalues

$$\frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - \frac{3}{4}\right)$$

$$= \frac{\hbar^2}{2}\left[(l\pm\tfrac{1}{2})(l+1\pm\tfrac{1}{2}) - l(l+1) - \tfrac{3}{4}\right]$$

$$= \begin{cases} \frac{1}{2}l\hbar^2 & \text{for } j = l+\frac{1}{2}, \\ -\frac{1}{2}(l+1)\hbar^2 & \text{for } j = l-\frac{1}{2}. \end{cases}$$

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(b) The subspaces to $j = \ell \pm \frac{1}{2}$ are also the subspaces in which $\frac{2}{\hbar^2}\vec{L}\cdot\vec{S}$ has values $\left\{ \begin{matrix} +\ell \\ -(\ell+1) \end{matrix} \right\}$, respectively. Therefore, the projectors are

$$P_+ = \frac{1}{2\ell+1}\left(\frac{2}{\hbar^2}\vec{L}\cdot\vec{S} + \ell + 1\right) \quad \text{for } j = \ell + \tfrac{1}{2}$$

and $$P_- = \frac{1}{2\ell+1}\left(\ell - \frac{2}{\hbar^2}\vec{L}\cdot\vec{S}\right) \quad \text{for } j = \ell - \tfrac{1}{2}.$$

(c) In state $|m_\ell, m_s\rangle$ we have $\langle L_z\rangle = \hbar m_\ell$ and $\langle S_z\rangle = \hbar m_s$ as well as $\langle L_x\rangle = \langle L_y\rangle = 0$ and $\langle S_x\rangle = \langle S_y\rangle = 0$, so that

$$\langle \vec{L}\cdot\vec{S}\rangle = \hbar^2 m_\ell m_s ,$$

and
$$\langle P_+\rangle = \frac{2 m_\ell m_s + \ell + 1}{2\ell+1}$$

is the resulting probability for $j = \ell + \frac{1}{2}$. For $m_\ell m_s = \ell \times \frac{1}{2} = (-\ell)\times(-\frac{1}{2})$ we have $\langle P_+\rangle = 1$, as we should have.

$\boxed{3}$ (a) We know, quite generally, that

$$\text{prob}(0\to 0, t) = 1 - \left(\frac{\delta H}{\hbar} t\right)^2$$

so that $\gamma = \frac{1}{\hbar}\delta H = \frac{1}{\hbar}\delta H_1$

$\delta H_1 = \langle H_1^2\rangle^{1/2}$ since $\langle H_1\rangle = 0$ here.

Now
$$\langle H_1^2\rangle = (\hbar\Omega)^2 \langle 0|(A^\dagger + A)^6|0\rangle$$
$$= (\hbar\Omega)^2 \langle 0|A(A^\dagger + A)^4 A^\dagger|0\rangle$$
$$= (\hbar\Omega)^2 \langle 1|(A^\dagger + A)^4|1\rangle ,$$

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where the relevant terms in $(A^\dagger + A)^4$ are

$$A^\dagger A^2 A^\dagger + A A^{\dagger 2} A + A^\dagger (A^\dagger A + A A^\dagger) A + A (A^\dagger A + A A^\dagger) A^\dagger$$

$$= 2(A^\dagger A)(A A^\dagger) + A^{\dagger 2} A^2 + A^2 A^{\dagger 2} + (A^\dagger A)^2 + (A A^\dagger)^2$$

which have expectation values

$$2 \cdot 1 \cdot 2 + 0 + 6 + 1^2 + 2^2 = 15$$

in state $|1\rangle = A^\dagger |0\rangle$. Thus $\langle H_1^2 \rangle = 15(\hbar\Omega)^2$ and

$$\gamma = \sqrt{15}\,\Omega .$$

(b) Since $e^{iH_0 t/\hbar} A^\dagger e^{-iH_0 t/\hbar} = e^{i\omega t A^\dagger A} A^\dagger e^{-i\omega t A^\dagger A}$

$$= e^{i\omega t} A^\dagger$$

and likewise $e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar} = e^{-i\omega t} A$

we get

$$\overline{H}_1(t) = \hbar\Omega \left(e^{i\omega t} A^\dagger + e^{-i\omega t} A\right)^3 .$$

(c) The probability amplitude for $0 \to n > 0$ is

$$-\frac{i}{\hbar} \int_0^T dt \, \langle n | \overline{H}_1(t) | 0 \rangle \quad \text{to first order in } \Omega$$

$$= -\frac{i}{\hbar} \int_0^T dt \, \langle n | \hbar\Omega \left(e^{i\omega t} A^\dagger + e^{-i\omega t} A\right)^3 | 0 \rangle$$

$$= -i\Omega \int_0^T dt \, \langle n | \left(e^{i\omega t} A^\dagger + e^{-i\omega t} A\right)^3 | 0 \rangle ,$$

where at most 3 operators $A^\dagger$ are acting on $|0\rangle$, so that we get zero except for $n = 1, 2, 3$.

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Further, there are 3 operators A^+ and A together, so that $|0\rangle$ is turned into a superposition of $|1\rangle$ and $|3\rangle$, but there is no amplitude for $|2\rangle$. In detail,

$$\left(A^+ e^{i\omega t} + A e^{-i\omega t}\right)^3 |0\rangle = |3\rangle \sqrt{6}\, e^{3i\omega t} + |1\rangle 3 e^{i\omega t},$$

and

$$p(0\to 1, T) = \left|\Omega \int_0^T dt\ 3\ e^{i\omega t}\right|^2$$

$$= \left(6 \frac{\Omega}{\omega} \sin\left(\frac{\omega T}{2}\right)\right)^2,$$

$$p(0\to 3, T) = \left|\Omega \int_0^T dt\ \sqrt{6}\ e^{3i\omega t}\right|^2$$

$$= \frac{8}{3}\left(\frac{\Omega}{\omega} \sin\left(\frac{3\omega T}{2}\right)\right)^2,$$

whereas $p(0\to n, T) = 0$ for $n=2$ and $n>3$ to lowest, ie. second, order in Ω.

$\boxed{4}$(a) We have

$$i\hbar \frac{\partial}{\partial t}(\alpha \psi_+ + \beta \psi_-) = \mathcal{H}(\alpha\psi_+ + \beta\psi_-)$$

$$= \hbar\omega(\alpha\psi_+ - \beta\psi_-)$$

from the Schrödinger equation, and

$$i\hbar \frac{\partial}{\partial t}(\alpha\psi_+ + \beta\psi_-) = i\hbar \frac{\partial\alpha}{\partial t}\psi_+ + i\hbar\frac{\partial\beta}{\partial t}\psi_- + i\hbar\,\alpha \frac{\partial\psi_+}{\partial t} + i\hbar\,\beta\frac{\partial\psi_-}{\partial t}$$

with $\frac{\partial}{\partial t}\psi_\pm = \frac{2\pi i}{T}\psi_\mp$ from the product rule

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and the explicit form of $\psi_I(t)$. Therefore,

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = i\mathcal{M}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

with
$$\mathcal{M} = \begin{pmatrix}-\omega & -2\pi/T\\ -2\pi/T & \omega\end{pmatrix}.$$

(b)
$$\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = e^{i\mathcal{M}t}\begin{pmatrix}\alpha(0)\\ \beta(0)\end{pmatrix}$$
$$= \left[\cos(\mathcal{M}t) + i\sin(\mathcal{M}t)\right]\begin{pmatrix}1\\ 0\end{pmatrix}$$

with $\cos(\mathcal{M}t) = \cos\left(\sqrt{\omega^2+(2\pi/T)^2}\,t\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

and $\sin(\mathcal{M}t) = \dfrac{\sin\left(\sqrt{\omega^2+(2\pi/T)^2}\,t\right)}{\sqrt{\omega^2+(2\pi/T)^2}}\mathcal{M}$,

because $\mathcal{M}^2 = (\omega^2+(2\pi/T)^2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Thus,

$$\alpha(t) = \cos\left(\sqrt{\cdot/\cdot}\,t\right) - \frac{i\omega}{\sqrt{\cdot/\cdot}}\sin\left(\sqrt{\cdot/\cdot}\,t\right),$$
$$\beta(t) = \frac{-2\pi i/T}{\sqrt{\cdot/\cdot}}\sin\left(\sqrt{\cdot/\cdot}\,t\right)$$

with $\sqrt{\cdot/\cdot} = \sqrt{\omega^2+(2\pi/T)^2} = \frac{1}{T}\sqrt{(2\pi)^2+(\omega T)^2}$.

(c) We have
$$|\beta(T)|^2 = \frac{(2\pi)^2}{(2\pi)^2+(\omega T)^2}\left[\sin\left(\sqrt{(2\pi)^2+(\omega T)^2}\right)\right]^2$$

where $\sqrt{(2\pi)^2+(\omega T)^2} \simeq \begin{cases} 2\pi + \frac{1}{4\pi}(\omega T)^2 & \text{for } \omega T \ll 1\\ \omega T & \text{for } \omega T \gg 1\end{cases}$

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so that

$$|\beta(T)|^2 \cong \left[\sin\left(\frac{(\omega T)^2}{4\pi}\right)\right]^2 \cong \frac{(\omega T)^4}{16\pi^2} \text{ for } \omega T \ll 1$$

and

$$|\beta(T)|^2 \cong \left(\frac{2\pi}{\omega T}\sin(\omega T)\right)^2 \propto \left(\frac{1}{\omega T}\right)^2 \text{ for } \omega T \gg 1.$$