Problem 1 (25 marks)

The periodic function $F(t) = F(t+T) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{-\mathrm{i}n\omega t}$ with $\omega T = 2\pi$ is specified by

$$F(t) = \cosh(\gamma t) \quad \text{with } \gamma > 0 \text{ for } -\frac{T}{2} \le t \le \frac{T}{2}\,.$$

Determine the Fourier coefficients f_n. Then consider $t = 0$ to establish the identity

$$\frac{x}{\sinh x} = 1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{x^2}{x^2 + (n\pi)^2}\,.$$

Which analogous statement do you obtain by considering $t = T/2$?

Problem 2 (15 marks)

The function $G(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t}\, g(\omega)$ is known to be real and odd,

$$G(t) = G(t)^* = -G(-t)\,.$$

What are the corresponding properties of $g(\omega)$ that are implied by these properties of $G(t)$?

Problem 3 (15 marks)

The four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are such that

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0\,.$$

How are $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})$, $\mathbf{c}\cdot(\mathbf{d}\times\mathbf{a})$, $\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})$ related to $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$?

Problem 4 (25 marks)

Scalar field $\Phi(\mathbf{r})$ is given by $\Phi(\mathbf{r}) = (\mathbf{b}\cdot\mathbf{r})^2$, whereby $\mathbf{b}$ is a constant vector that does not depend on the position vector $\mathbf{r}$.

(a) Determine the gradient of $\Phi(\mathbf{r})$.

(b) Determine the divergence of the gradient of $\Phi(\mathbf{r})$.

(c) Convert the surface integral

$$\int_{S_R} \mathrm{d}\mathbf{S}\cdot\nabla\Phi$$

into a volume integral, whereby S_R is the sphere of radius R, centered at $\mathbf{r} = 0$. Evaluate explicitly both this surface integral and the equivalent volume integral. [Hint: You may choose a convenient direction for $\mathbf{b}$.]

Problem 5 (20 marks)

$a(\mathbf{r})$ is a scalar field; $\mathbf{B}(\mathbf{r})$ and $\mathbf{C}(\mathbf{r})$ are vector fields. Express

$$\nabla\times[a(\mathbf{r})\mathbf{B}(\mathbf{r})\times\mathbf{C}(\mathbf{r})]$$

as a sum of three terms, such that only $a(\mathbf{r})$ is differentiated in the first term, only $\mathbf{B}(\mathbf{r})$ in the second term, and only $\mathbf{C}(\mathbf{r})$ in the third term.