Question ....

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{S1}$ $\frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v = 2x-2y$, $\frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} v = -2x-2y$

& $u(0,0)=0$ imply $u(x,y) = x^2-2xy-y^2$.
Compactly: $f(z) = (1+i)\,z^2$.

$\boxed{S2}$ $\int_{-\infty}^{\infty} dx\, \delta(x^2-4x+3)\, e^{i\pi x} = \int_{-\infty}^{\infty} dx\, \delta\big((x-1)(x-3)\big)\, e^{i\pi x}$

$= \int_{-\infty}^{\infty} dx\, \left(\tfrac{1}{2}\delta(x-1) + \tfrac{1}{2}\delta(x-3)\right) e^{i\pi x} = \tfrac{1}{2} e^{i\pi} + \tfrac{1}{2} e^{3i\pi}$

$= -1.$

$\boxed{S3}$ $\Phi = b^2 r^2 - (\vec{b}\cdot\vec{r})^2$, $\vec{\nabla}\Phi = 2b^2\vec{r} - 2\vec{b}\cdot\vec{r}\;\vec{b}$
$= 2\,\vec{b}\times(\vec{r}\times\vec{b})$;
$\vec{\nabla}^2\Phi = 2b^2\,\vec{\nabla}\cdot\vec{r} - 2\vec{b}\cdot\vec{\nabla}\;\vec{b}\cdot\vec{r} = 6b^2 - 2b^2 = 4b^2.$

$\boxed{S4}$ $\int_S d\vec{S}\;\vec{\nabla}\cdot\vec{F} = \int_V (d\vec{r})\;\vec{\nabla}\,\vec{\nabla}\cdot\vec{F}$

$= \int_V (d\vec{r})\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) + \vec{\nabla}^2\vec{F}\right].$

$\boxed{S5}$ $y\frac{dy}{dx} = x \;\leadsto\; \frac{d}{dx}(y^2-x^2) = 0$, so that
$y(x) = \sqrt{9+x^2}$ for $y(0)=3$. Then
$y(4) = \sqrt{9+16} = 5.$

Question ....

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{L1}$ $F(t)$ has 2nd-order poles at $t = \pm iT$. Therefore,

$$f(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \frac{1}{(t^2+T^2)^2}$$

$$= \int_{-\infty}^{\infty} dt\, e^{i|\omega|t} \frac{1}{(t^2+T^2)^2}$$

$$= 2\pi i \times \left\{ \text{residue of } \frac{e^{i|\omega|t}}{(t^2+T^2)^2} \text{ at } t=+iT \right\}$$

$$= 2\pi i \left. \frac{d}{dt} \frac{e^{i|\omega|t}}{(t+iT)^2} \right|_{t=iT}$$

$$= 2\pi i \left(\frac{i|\omega|\, e^{-|\omega|T}}{(2iT)^2} - 2\frac{e^{-|\omega|T}}{(2iT)^3} \right)$$

$$= \frac{2\pi i}{(2iT)^3} \left(2iT\, i|\omega| - 2\right) e^{-|\omega|T}$$

$$= \frac{\pi}{2T^3} \left(1 + |\omega|T\right) e^{-|\omega|T} .$$

Verify that $f(\omega=0) = \int_{-\infty}^{\infty} dt\, F(t)$:

(1) $f(0) = \frac{\pi}{2T^3}$ from above

(2) $\int_{-\infty}^{\infty} dt\, F(t) = -\frac{1}{2T}\frac{\partial}{\partial T} \underbrace{\int_{-\infty}^{\infty} \frac{dt}{t^2+T^2}}_{=\frac{\pi}{T}} = \frac{\pi}{2T^3}$.

Indeed, $f(\omega=0)$ has the correct value.

Question ....

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{L2}$ (a) $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) = [(\vec{a}\times\vec{b})\cdot\vec{c}]\,\vec{b}$, so that

$$[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})]\cdot(\vec{c}\times\vec{a}) = [(\vec{a}\times\vec{b})\cdot\vec{c}]^2 .$$

(b) $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$, $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$.

(c) $\vec{a}' \mathrel{\hat{=}} \frac{1}{2}\begin{pmatrix}-1\\1\\1\end{pmatrix}$, $\vec{b}' \mathrel{\hat{=}} \frac{1}{2}\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\vec{c}' \mathrel{\hat{=}} \frac{1}{2}\begin{pmatrix}1\\1\\-1\end{pmatrix}$

either by construction according to (b) or by inspection.

$\boxed{L3}$ (a) $G(t,\pi-\varphi)^* = \sum_m i^{-m} e^{-im(\pi-\varphi)} J_m(t)^*$

$$= \sum_m i^m e^{im\varphi} J_m(t)^*$$

so that $J_m(t)^* = J_m(t)$;

$$G(t,-\varphi) = \sum_m i^m e^{-im\varphi} J_m(t)$$

$$= \sum_m i^m e^{im\varphi} (-1)^m J_{-m}(t)$$

so that $J_m(t) = (-1)^m J_{-m}(t)$;

$$G(-t,\pi+\varphi) = \sum_m i^m e^{im(\pi+\varphi)} J_m(-t)$$

$$= \sum_m i^m e^{im\varphi} (-1)^m J_m(-t)$$

so that $J_m(t) = (-1)^m J_m(-t)$.

Question ....

Write answers on this side of the paper only.

Do not write on either margin

(b) $\frac{\partial}{\partial\varphi} G(t,\varphi) = -it\sin\varphi\, G(t,\varphi)$

$$= \frac{t}{2}\sum_m i^m \left(-e^{i(m+1)\varphi} + e^{i(m-1)\varphi}\right) J_m(t)$$

$$= \frac{t}{2}\sum_m i^m e^{im\varphi}\left(i J_{m-1}(t) + i J_{m+1}(t)\right)$$

$$= \frac{\partial}{\partial\varphi}\sum_m i^m e^{im\varphi} J_m(t) = \sum_m i^m e^{im\varphi}\, im\, J_m(t)$$

gives $\frac{2m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t)$;

$\frac{\partial}{\partial t} G(t,\varphi) = i\cos\varphi\, G(t,\varphi)$

$$= \frac{1}{2}\sum_m i^m \left(ie^{i(m+1)\varphi} + ie^{i(m-1)\varphi}\right) J_m(t)$$

$$= \frac{1}{2}\sum_m i^m e^{im\varphi}\left(J_{m-1}(t) - J_{m+1}(t)\right)$$

$$= \frac{\partial}{\partial t}\sum_m i^m e^{im\varphi} J_m(t) = \sum_m i^m e^{im\varphi}\frac{d}{dt}J_m(t)$$

gives $2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t)$.