These sample solutions were prepared by Bess Fang.

Note: All comments within square brackets refer to the previous line.

Problem 1

We compute $|1\rangle\langle 1|$, $|2\rangle\langle 2|$, $|3\rangle\langle 3|$:

$$|1\rangle\langle 1| \widehat{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$|2\rangle\langle 2| \widehat{=} \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \frac{1}{5} (3, 4) = \frac{1}{25} \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix},$$

$$|3\rangle\langle 3| \widehat{=} \frac{1}{5} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \frac{1}{5} (3, -4) = \frac{1}{25} \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix}.$$

Substituting these into

$$\rho = |1\rangle w_1 \langle 1| + |2\rangle w_2 \langle 2| + |3\rangle w_3 \langle 3| \widehat{=} \frac{1}{20} \begin{pmatrix} 12 & 3 \\ 3 & 8 \end{pmatrix}$$

results in the set of equations

$$\left\{ \begin{aligned} w_1 + \frac{9}{25} w_2 + \frac{9}{25} w_3 &= \frac{12}{20} \\ \frac{12}{25} w_2 - \frac{12}{25} w_3 &= \frac{3}{20} \\ \frac{16}{25} w_2 + \frac{16}{25} w_3 &= \frac{8}{20} \end{aligned} \right\} \quad \text{which yield} \quad \left\{ \begin{aligned} w_1 &= \frac{3}{8} = \frac{12}{32}, \\ w_2 &= \frac{15}{32}, \\ w_3 &= \frac{5}{32}. \end{aligned} \right.$$

As expected, the weights have unit sum, $w_1 + w_2 + w_3 = 1$.

Problem 2

We compute

$$
\begin{aligned}
\langle p|F|x\rangle &= \int dx'\ dp'\ \langle p|x'\rangle\ \langle x'|F|p'\rangle\ \langle p'|x\rangle \\
&= \int_{-\infty}^{\infty} dx'\ dp'\ \frac{e^{-ipx'/\hbar}}{\sqrt{2\pi\hbar}}\ e^{-(a|x'|+b|p'|)/\hbar}\ \frac{e^{-ip'x/\hbar}}{\sqrt{2\pi\hbar}} \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx'\ e^{-(ipx'+a|x'|)/\hbar}\ \int_{-\infty}^{\infty} dp'\ e^{-(ip'x+b|p'|)/\hbar}.
\end{aligned}
$$

Since the two integrals are of exactly the same form, we only need to evaluate one of them:

$$
\begin{aligned}
& \int_{-\infty}^{\infty} dx'\ e^{-(ipx'+a|x'|)/\hbar} \\
&= \int_{-\infty}^{0} dx'\ e^{(-ipx'+ax')/\hbar} + \int_{0}^{\infty} dx'\ e^{(-ipx'-ax')/\hbar} \\
&= \int_{+\infty}^{0} d(-x')\ e^{(-ip(-x')+a(-x'))/\hbar} + \int_{0}^{\infty} dx'\ e^{(-ipx'-ax')/\hbar} \\
&\quad \text{[replacing } x' \text{ by } -x'\text{, note the changed limits]} \\
&= \int_{0}^{\infty} dx'\ e^{(ip-a)x'/\hbar} + \int_{0}^{\infty} dx'\ e^{(-ip-a)x'/\hbar} \\
&= \frac{\hbar}{a-ip} + \frac{\hbar}{a+ip} = \frac{2a\hbar}{a^2+p^2}.
\end{aligned}
$$

Likewise, we have for the 2nd integral

$$
\int_{-\infty}^{\infty} dp'\ e^{-(ip'x+b|p'|)/\hbar} = \frac{2b\hbar}{b^2+x^2}.
$$

Therefore,

$$
\langle p|F|x\rangle = \frac{2\hbar}{\pi}\ \frac{a}{a^2+p^2}\ \frac{b}{b^2+x^2}.
$$

Problem 3

For the position wave function $\langle x|\lambda\rangle$, the eigenket equation $\Lambda|\lambda\rangle = |\lambda\rangle\lambda$ reads

$$\langle x|\Lambda|\lambda\rangle = \langle x|\lambda\rangle\lambda = \frac{1}{\hbar}\big(p_0 x + x_0 \frac{\hbar}{i}\frac{\partial}{\partial x}\big)\langle x|\lambda\rangle$$

or, after a simple rearrangement,

$$x_0 \, \frac{\partial}{\partial x} \, \log\langle x|\lambda\rangle = -i\Big(\frac{x}{x_0} - \lambda\Big) .$$

This differential equation is solved by

$$\langle x|\lambda\rangle = C(\lambda) \, e^{-\frac{i}{2}(x/x_0-\lambda)^2} ,$$

where $C(\lambda)$ is the multiplicative integration constant that could depend on λ, but not on x.

Now, we take care of the normalization by considering $\langle\lambda|\lambda'\rangle$:

$$\begin{aligned}
\langle\lambda|\lambda'\rangle &= \int dx \, \langle\lambda|x\rangle\langle x|\lambda'\rangle \\
&= \int dx \, C(\lambda)^* e^{\frac{i}{2}(x/x_0-\lambda-)^2} \, C(\lambda') e^{-\frac{i}{2}(x/x_0-\lambda')^2} \\
&= C(\lambda)^* C(\lambda') \, e^{\frac{i}{2}(\lambda^2-\lambda'^2)} \int dx \, e^{i\frac{x}{x_0}(\lambda'-\lambda)} \\
&= C(\lambda)^* C(\lambda') \, e^{\frac{i}{2}(\lambda^2-\lambda'^2)} \, 2\pi x_0 \, \delta(\lambda'-\lambda) \\
&= |C(\lambda)|^2 \, 2\pi x_0 \, \delta(\lambda'-\lambda) .
\end{aligned}$$

[the exponential factor becomes 1 when $\lambda = \lambda'$, and $C(\lambda')$ becomes $C(\lambda)$]

Since we require $\langle\lambda|\lambda'\rangle = \delta(\lambda'-\lambda)$, we have

$$|C(\lambda)|^2 \, 2\pi x_0 = 1, .$$

Taking $C(\lambda) > 0$,

$$C(\lambda) = C = \frac{1}{\sqrt{2\pi x_0}} .$$

so that, finally,

$$\langle x|\lambda\rangle = \frac{1}{\sqrt{2\pi x_0}} e^{-\frac{i}{2}(x/x_0-\lambda)^2} .$$

Problem 4

(a) We have the Heisenberg equations of motion

$$\frac{dP}{dt} = -\frac{\partial H}{\partial X} = F \qquad \text{and} \qquad \frac{dX}{dt} = \frac{\partial H}{\partial P} = \frac{P}{M}\,,$$

which are solved by

$$P(t) = P(t_0) + FT\,, \tag{1}$$

$$X(t) = X(t_0) + \frac{T}{M}P(t_0) + \frac{FT^2}{2M}\,, \tag{2}$$

where $T = t - t_0$. Then

$$\begin{aligned}
[X(t), X(t_0)] &= \left[X(t_0) + \frac{T}{M}P(t_0) + \frac{FT^2}{2M}, X(t_0)\right] \\
&= \left[\frac{T}{M}P(t_0), X(t_0)\right] \\
&\quad \text{[since } X(t_0) \text{ commutes with itself and the numerical term]} \\
&= -i\hbar\frac{T}{M}\,.
\end{aligned}$$

(b) First we use Eqs. (1) and (2) to express $P(t_0)$ and $P(t)$ in terms of $X(t_0)$ and $X(t)$,

$$\begin{aligned}
P(t_0) &= \frac{M}{T}\big(X(t) - X(t_0)\big) - \frac{1}{2}FT\,, \\
P(t) &= \frac{M}{T}\big(X(t) - X(t_0)\big) + \frac{1}{2}FT\,,
\end{aligned}$$

Then we use these in

$$\begin{aligned}
i\hbar\frac{\partial}{\partial x}\langle x, t|x', t_0\rangle &= -\langle x, t|P(t)|x', t_0\rangle \\
&= \Big(-\frac{M}{T}(x - x') - \frac{1}{2}FT\Big)\langle x, t|x', t_0\rangle
\end{aligned}$$

and

$$\begin{aligned}
i\hbar\frac{\partial}{\partial x'}\langle x, t|x', t_0\rangle &= \langle x, t|P(t_0)|x', t_0\rangle \\
&= \Big(\frac{M}{T}(x - x') - \frac{1}{2}FT\Big)\langle x, t|x', t_0\rangle\,.
\end{aligned}$$

Now we divide by $\langle x,t|x',t_0\rangle$ and arrive at

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial x}\log\langle x,t|x',t_0\rangle &= -\frac{M}{T}(x-x')-\frac{1}{2}FT\,,\\
i\hbar\frac{\partial}{\partial x'}\log\langle x,t|x',t_0\rangle &= \frac{M}{T}(x-x')-\frac{1}{2}FT\,.
\end{aligned}
$$

Next, we need to express the Hamilton operator as a function of $X(t)$ and $X(t_0)$ with $X(t)$ to the left of $X(t_0)$ in products,

$$
\begin{aligned}
H &= \frac{1}{2M}P(t)^2 - FX(t)\\
&= \frac{1}{2M}\left[\frac{M}{T}\big(X(t)-X(t_0)\big)+\frac{1}{2}FT\right]^2 - FX(t)\\
&= \frac{M}{2T^2}\Big[X(t)^2+X(t_0)^2-X(t)X(t_0)-X(t_0)X(t)\Big]\\
&\quad -\frac{F}{2}\big[X(t)+X(t_0)\big]+\frac{F^2T^2}{8M}\\
&= \frac{M}{2T^2}\Big[X(t)^2+X(t_0)^2-2X(t)X(t_0)-i\hbar\frac{T}{M}\Big]\\
&\quad -\frac{F}{2}\big[X(t)+X(t_0)\big]+\frac{F^2T^2}{8M}
\end{aligned}
$$

[we are using the commutator found in **(a)**]

which we now use in

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\langle x,t|x',t_0\rangle &= \langle x,t|H|x',t_0\rangle\\
&= \left(\frac{M}{2T^2}\Big[(x-x')^2-i\hbar\frac{T}{M}\Big]-\frac{F}{2}(x+x')+\frac{F^2T^2}{8M}\right)\\
&\quad\times\langle x,t|x',t_0\rangle\,.
\end{aligned}
$$

Thus, we have

$$
i\hbar\frac{\partial}{\partial t}\;\log\langle x,t|x',t_0\rangle = \frac{M}{2T^2}(x-x')^2-\frac{F}{2}(x+x')+\frac{F^2T^2}{8M}-\frac{i\hbar}{2T}\,.
$$