Problem 1 (20 marks)
A harmonic oscillator (mass M, natural frequency ω) is in its ground state. Determine the expectation value of $\frac{1}{2}[X(t_1)X(t_2) + X(t_2)X(t_1)]$ for any two times t_1 and t_2.

Problem 2 (20 marks)
Orbital angular momentum: If $\vec{L}^2$ has the value $l(l+1)\hbar^2$, with $l = 0, 1, 2, \dots$, what is the value of

$$\mathrm{tr}\left\{\mathrm{e}^{\gamma L_3/\hbar}\right\}$$

for real γ?

Problem 3 (30 marks)
A hydrogenic atom (as usual: electron mass M, electron charge $-e$, nuclear charge Ze) is exposed to a perturbing potential that is given by

$$H_1 = \frac{V_0}{(r/a_0)^2} \,,$$

where $V_0 > 0$ is the strength of the perturbation and $a_0 = \dfrac{\hbar^2}{Me^2}$ is the Bohr radius. What is the energy of a bound state with radial quantum number n_r and angular momentum quantum number l? [Hint: You can state the exact energy eigenvalues after considering the radial Schrödinger equation.]

Problem 4 (30 marks)
Motion along the x axis; mass M, position operator X, momentum operator P. Use trial wave functions of the form $\psi(x) = \sqrt{\kappa}\,\mathrm{e}^{-\kappa|x|}$, with an adjustable parameter $\kappa > 0$, to establish upper bounds on the ground state energy of the Hamilton operator

$$H = \frac{1}{2M}P^2 - \frac{(\hbar\kappa_0)^2}{M}\mathrm{e}^{-\kappa_0|X|} \,,$$

where $\kappa_0 > 0$ specifies the strength and the range of the potential energy. For which value of κ do you get the best upper bound? What is its value?