Question ....

Write answers on this side of the paper only.

Do not write on either margin

1 (a) $\frac{d}{dt} P(t) = \omega \left(P - M\omega X\right)$, $\frac{d}{dt} X(t) = \frac{1}{M}\left(P - M\omega X\right)$;

$P(t) - M\omega X(t) = P(t_0) - M\omega X(t_0)$ is constant.

$$P(t) = P(t_0) + \omega T \left(P - M\omega X\right)(t_0)$$
$$= (1+\omega T)\, P(t_0) - M\omega^2 T\, X(t_0);$$

$$X(t) = X(t_0) + \frac{T}{M}\left(P - M\omega X\right)(t_0)$$
$$= (1-\omega T)\, X(t_0) + \frac{T}{M} P(t_0) \quad \text{with } T = t - t_0.$$

$$[X(t), X(t_0)] = \frac{T}{M}\left[P(t_0), X(t_0)\right] = -i\hbar \frac{T}{M}.$$

(b) Use $X(t) = X(t_0) + \frac{T}{M}\left(P - M\omega X\right)(t)$ for

$P(t) = \frac{M}{T}\left[(1+\omega T) X(t) - X(t_0)\right]$, and from above

$P(t_0) = \frac{M}{T}\left[X(t) - (1-\omega T) X(t_0)\right]$,

$P(t) - M\omega X(t) = \frac{M}{T}\left(X(t) - X(t_0)\right)$, so that

$$H = \frac{M}{2T^2}\left(X(t) - X(t_0)\right)^2$$
$$= \frac{M}{2T^2}\left(X(t)^2 - 2X(t)X(t_0) + X(t_0)^2 + [X(t), X(t_0)]\right)$$
$$= \frac{M}{2T^2}\left(X(t)^2 - 2X(t)X(t_0) + X(t_0)^2 - i\hbar\frac{T}{M}\right).$$

(c) $\frac{\partial}{\partial x}\langle x, t | x', t_0\rangle = \frac{i}{\hbar}\langle x, t | P(t) | x', t_0\rangle$,

$\frac{\partial}{\partial x'}\langle x, t | x', t_0\rangle = -\frac{i}{\hbar}\langle x, t | P(t_0) | x', t_0\rangle$,

Question ....

Write answers on this side of the paper only.

Do not write on either margin

and $\frac{\partial}{\partial T}\langle x,t|x',t_0\rangle = \frac{\partial}{\partial t}\langle x,t|x',t_0\rangle$

$$= -\frac{i}{\hbar}\langle x,t|H(t)|x',t_0\rangle$$

give

$$\frac{\partial}{\partial x}\log\langle x,t|x',t_0\rangle = \frac{i}{\hbar}\left(\frac{M}{T}(x-x') + M\omega x\right),$$

$$\frac{\partial}{\partial x'}\log\langle x,t|x',t_0\rangle = \frac{i}{\hbar}\left(-\frac{M}{T}(x-x') - M\omega x'\right),$$

$$\frac{\partial}{\partial T}\log\langle x,t|x',t_0\rangle = \frac{i}{\hbar}\left(-\frac{M}{2T^2}(x-x')^2\right) - \frac{1}{2T}$$

so that

$$\log\langle x,t|x',t_0\rangle = \frac{i}{\hbar}\frac{M}{2T}(x-x')^2 + \log\frac{1}{\sqrt{T}} + \frac{i}{\hbar}\frac{M\omega}{2}(x^2-x'^2) + \text{const.}$$

The constant is determined by the known result for $\omega=0$, giving

$$\langle x,t|x',t_0\rangle = \sqrt{\frac{M}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}\frac{M}{2T}(x-x')^2} e^{\frac{i}{\hbar}\frac{M\omega}{2}(x^2-x'^2)}.$$

Question ....

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{2}$(a) To find $\langle a^*|Z|a'\rangle$ we consider

$$\langle a^*|ZA^\dagger|a'\rangle = \frac{\partial}{\partial a'}\langle a^*|Z|a'\rangle$$

$$= (1-\lambda)\langle a^*|A^\dagger Z|a'\rangle = (1-\lambda)a^*\langle a^*|Z|a'\rangle$$

and the adjoint statement $AZ=(1-\lambda)ZA$, which gives

$$\frac{\partial}{\partial a^*}\langle a^*|Z|a'\rangle = (1-\lambda)a'\langle a^*|Z|a'\rangle.$$

Accordingly, $\langle a^*|Z|a'\rangle = \underbrace{C}_{\text{constant}} e^{(1-\lambda)a^*a'}$.

$$= \langle a^*|a'\rangle\, C\, e^{-\lambda a^*a'},$$

so that

$$Z = C\, e^{-\lambda A^\dagger;A}.$$

We use the 2nd parameterization for the evaluation of the trace,

$$1 = \mathrm{tr}\{Z\} = C\int\frac{dx\,dp}{2\pi\hbar}\, e^{-\lambda(-i\ell p/\hbar)(x/\ell)}$$

$$= C\int dx\,\delta(\lambda x) = \frac{1}{\lambda}C;$$

thus $C=\lambda$ and

$$Z = \lambda\, e^{-\lambda A^\dagger;A}.$$

Question ....

Write answers on this side of the paper only.

Do not write on either margin

(b) $Z A^+ A = (1-\lambda) A^+ Z A = A^+ A Z$.

Write $Z = f(A^+A)$, then

$$Z A^+ = A^+ f(A^+A+1) = (1-\lambda) A^+ f(A^+A)$$

or

$$f(A^+A+1) = (1-\lambda) f(A^+A),$$

implying

$$f(A^+A) = (\text{const}) \, (1-\lambda)^{A^+A},$$

or

$$Z = \tilde{C} (1-\lambda)^{A^+A}, \quad \tilde{C} = \text{const.}$$

Now use Fock state for the trace

$$1 = \mathrm{tr}\{Z\} = \tilde{C} \sum_{n=0}^{\infty} (1-\lambda)^n = \frac{\tilde{C}}{\lambda};$$

thus $\tilde{C} = \lambda$ and

$$Z = \lambda (1-\lambda)^{A^+A} .$$

Do not write on either margin

$\boxed{3}$ (a) (i) & (ii) outcomes are $0\hbar^2, 1\hbar^2, 4\hbar^2$ for $m = 0, \pm 1, \pm 2$, the m values to $l = 2$, for which $l(l+1) = 6$.

(iii) $L_1^2 + L_2^2 = \vec{L}^2 - L_3^2$, so that the outcomes are $(6-0)\hbar^2, (6-1)\hbar^2, (6-4)\hbar^2$, that is $6\hbar^2, 5\hbar^2, 2\hbar^2$.

(b) Since $L_1^2 - L_2^2 = \frac{1}{2}(L_+^2 + L_-^2)$ we have for $|l,m\rangle \equiv |m\rangle$ eigenket for L_3 and $\vec{L}^2$:
($l=2$ $m = 0, \pm 1, \pm 2$ here)

$$(L_1^2 - L_2^2)|1\rangle = \frac{1}{2}(L_+^2 + L_-^2)|1\rangle = \frac{1}{2}L_-^2|1\rangle$$
$$= |-1\rangle \frac{\hbar^2}{2}\sqrt{(2+1)(3-1)}\sqrt{(2+0)(3-0)}$$
$$= |-1\rangle 3\hbar^2,$$

likewise

$$(L_1^2 - L_2^2)|-1\rangle = |1\rangle 3\hbar^2,$$

so that $(L_1^2 - L_2^2)(|1\rangle \pm |-1\rangle) = (|1\rangle \pm |-1\rangle)(\pm 3\hbar^2)$ which identifies the eigenvalues $\pm 3\hbar^2$.

Further

$$(L_1^2 - L_2^2)|2\rangle = (L_1^2 - L_2^2)|-2\rangle = |0\rangle\sqrt{6}\,\hbar^2$$

and

$$(L_1^2 - L_2^2)|0\rangle = (|2\rangle + |-2\rangle)\sqrt{6}\,\hbar^2$$

so that

$$(L_1^2 - L_2^2)(|2\rangle - |-2\rangle) = 0 \quad \text{and}$$

$$(L_1^2 - L_2^2)(|2\rangle \pm |0\rangle\sqrt{2} + |-2\rangle)$$
$$= (|2\rangle \pm |0\rangle\sqrt{2} + |-2\rangle)(\pm\sqrt{12}\,\hbar^2)$$

which identify the eigenvalues $0, \pm\sqrt{12}\,\hbar^2$.

Question ....

Write answers on this side of the paper only.

Do not write on either margin

In summary, the possible outcomes are
$0\hbar^2, \ \pm 3\hbar^2, \ \pm\sqrt{12}\,\hbar^2$
when $L_1^2 - L_2^2$ is measured.

(c) $\langle L_1 \rangle = \langle L_2 \rangle = 0$ because $L_\pm = L_1 \pm iL_2$
and $\langle L_+ \rangle = 0, \ \langle L_- \rangle = 0$;

$$\langle L_1^2 \rangle = \langle L_2^2 \rangle = \tfrac{1}{2}\langle (L_1^2 + L_2^2)\rangle \ \text{(symmetry)}$$
$$= \tfrac{1}{2}\langle (\vec{L}^2 - L_3^2)\rangle$$
$$= \tfrac{1}{2}\left(6\hbar^2 - (m\hbar)^2\right)$$
$$= \left(3 - \tfrac{1}{2}m^2\right)\hbar^2 .$$

Spreads: $\delta L_1 = \delta L_2 = \sqrt{3 - \tfrac{1}{2}m^2}\,\hbar$

$$= \begin{cases} \sqrt{3}\,\hbar & \text{for } m = 0, \\ \sqrt{5/2}\,\hbar & \text{for } m = \pm 1, \\ \hbar & \text{for } m = \pm 2. \end{cases}$$

Question ....

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{4}$ (a) For $\xi = 0$:
$$e^{-t^2} = \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{m!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(0)$$

so that $H_n(0) = 0$ for n odd,
$$H_n(0) = (-1)^m \frac{(2m)!}{m!} \text{ for } n = 2m \text{ even.}$$

Recall
$$\langle x|n\rangle = \pi^{-1/4} \ell^{-1/2} \frac{1}{\sqrt{2^n n!}} e^{-\frac{1}{2}\left(\frac{x}{\ell}\right)^2} H_n\left(\frac{x}{\ell}\right),$$

so
$$\langle x|n\rangle\Big|_{x=0} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \pi^{-1/4} \ell^{-1/2} (-1)^m \frac{\sqrt{(2m)!}}{2^m\, m!} & \text{for } n = 2m \text{ even} \end{cases}$$

(b) Use the Hellmann - Feynman Theorem to get
$$\frac{\partial E_n}{\partial V}\Big|_{V=0} = \underbrace{\sqrt{\frac{\hbar}{M\omega}}}_{=\ell} \underbrace{\langle n|\delta(x)|n\rangle}_{=\left|\langle x|n\rangle|_{x=0}\right|^2}$$

$$= \begin{cases} \frac{1}{\sqrt{\pi}} \frac{1}{4^m} \frac{(2m)!}{(m!)^2} = \frac{1}{\sqrt{\pi}} \frac{1}{4^m} \binom{2m}{m}. \\ \quad \text{for } n = 2m \text{ even,} \\ 0 \quad \text{for } n \text{ odd.} \end{cases}$$

(c) large even n:
$$\frac{\partial E_n}{\partial V}\Big|_{V=0} \cong \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi n/2}} = \frac{1}{\pi}\sqrt{\frac{2}{n}};$$

large odd n: $\frac{\partial E_n}{\partial V} = 0$.