Question. Test 1/1

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$\boxed{1}$ The characteristic curves are determined by

$$\begin{pmatrix} x \\ -y \\ 2z \end{pmatrix} \propto \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \text{ or } \frac{dy}{dx} = -\frac{y}{x}$$

$$\text{and } \frac{dz}{dx} = \frac{2z}{x}.$$

Therefore

$$\frac{dx}{x} + \frac{dy}{y} = 0 \quad \text{or} \quad xy = \text{const}$$

and

$$\frac{dx}{x} - \frac{dz}{2z} = 0 \quad \text{or} \quad z/x^2 = \text{const},$$

which imply $z(x,y) = x^2\, u(xy)$ for the general solution. With $z(x,x) = 1$, we have $x^2 u(x^2) = 1$, so that $u(xy) = \frac{1}{xy}$, and the special solution is

$$z(x,y) = \frac{x}{y}.$$

$\boxed{2}$ Since $\frac{d}{d\varphi} \tan\varphi = \left(\frac{1}{\cos\varphi}\right)^2 = 1 + (\tan\varphi)^2$, we have

$$\frac{\partial}{\partial x} z(x,y) = \left[1 + z(x,y)^2\right]\left(1 + 2x\, u'(x^2+y)\right),$$

$$\frac{\partial}{\partial y} z(x,y) = \left[1 + z(x,y)^2\right] u'(x^2+y),$$

so that

$$\left(\frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y}\right) z(x,y) = 1 + z(x,y)^2$$

after eliminating the derivative $u'(\ )$ of the arbitrary function $u(\ )$.

Question. Test 1/2

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$\boxed{3}$ The Hamilton equations of motion are

$$\frac{d}{dt}x = \frac{\partial H}{\partial p} = \frac{1}{m}p \ , \quad \frac{d}{dt}p = -\frac{\partial H}{\partial x} = m\gamma^2 x,$$

which are solved by

$$x(t) = x(0)\cosh(\gamma t) + \frac{p(0)}{m\gamma}\sinh(\gamma t),$$

$$p(t) = p(0)\cosh(\gamma t) + m\gamma x(0)\sinh(\gamma t).$$

We solve for $x(0)$ and $p(0)$,

$$x(0) = x(t)\cosh(\gamma t) - \frac{p(t)}{m\gamma}\sinh(\gamma t),$$

$$p(0) = p(t)\cosh(\gamma t) - m\gamma x(t)\sinh(\gamma t),$$

and thus obtain

$$\rho(t,x,p) = \rho_0\left(x\cosh(\gamma t) - \frac{p}{m\gamma}\sinh(\gamma t), p\cosh(\gamma t) - m\gamma x\sinh(\gamma t)\right).$$

$\boxed{4}$ We have

$$\int_0^1 dx\ \delta y\left(-2\frac{d^2}{dx^2}y\right) = 0$$

with the constraint

$$\int_0^1 dx\ \delta y = 0$$

so that $\frac{d^2}{d^2x}y = \lambda = \text{const}.$

Question. Test 1/3

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This gives $y(x) = -\frac{\lambda}{2}\, x(1-x)$, so that the value of the Lagrange multiplier is determined by

$$2 = \int_0^1 dx \left(-\frac{\lambda}{2}\right)(x-x^2) = -\frac{\lambda}{4} + \frac{\lambda}{6} = -\frac{\lambda}{12}$$

or $\lambda = -24$. Therefore, $y = 12\,(x-x^2)$

and $$\int_0^1 dx \left(\frac{dy}{dx}\right)^2 = \int_0^1 dx\,\left[12\,(1-2x)\right]^2$$

$$= 144 \int_0^1 dx\,(1-4x+4x^2) = 144\left(1-\frac{4}{2}+\frac{4}{3}\right)$$

$$= 48.$$ Answer: The smallest value is 48.

$\boxed{5}$ With $x = s\cos\varphi$, $y = s\sin\varphi$, $z = \frac{1}{2}\kappa(x^2+y^2) = \frac{1}{2}\kappa s^2$,

we have

$$\dot{x} = \dot{s}\cos\varphi - s\dot{\varphi}\sin\varphi\,,$$
$$\dot{y} = \dot{s}\sin\varphi + s\dot{\varphi}\cos\varphi\,,$$
$$\dot{z} = \kappa s\dot{s}\,,$$

and get $\frac{m}{2}\left[\dot{s}^2 + (s\dot{\varphi})^2 + (\kappa s\dot{s})^2\right]$ for the kinetic energy. The potential energy is mgz, and so the Lagrange function is

$$L(s,\varphi,\dot{s},\dot{\varphi}) = \frac{m}{2}(1+\kappa^2 s^2)\dot{s}^2 + \frac{m}{2}s^2\dot{\varphi}^2 - \frac{1}{2}mg\kappa s^2.$$

Question. Test 1/4

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This gives

$$p_s = \frac{\partial L}{\partial \dot{s}} = m(1+k^2 s^2)\dot{s},$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m s^2 \dot{\varphi}$$

for the canonical momenta, and then

$$H = p_s \dot{s} + p_\varphi \dot{\varphi} - L \quad \left(\begin{array}{l}\text{with } \dot{s}, \dot{\varphi} \text{ eliminated} \\ \text{in favor of } p_s \text{ and } p_\varphi\end{array}\right)$$

$$= \frac{p_s^2}{2m(1+k^2 s^2)} + \frac{p_\varphi^2}{2ms^2} + \frac{1}{2} mgk s^2$$

for the Hamilton function $H(p_s, p_\varphi, s, \varphi)$, in which φ is a cyclic variable.