Problem 1 (15 marks)
Function $\mathsf{z}(x,y)$ obeys the quasi-linear partial differential equation (qlPDE)

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\mathsf{z}(x,y) = 2\mathsf{z}(x,y)\,.$$

Determine the general solution of this qlPDE and the special solution for $\mathsf{z}(x,x) = 1$.

Problem 2 (15 marks)
The general solution of a certain qlPDE for $\mathsf{z}(x,y)$ is of the form

$$\mathsf{z}(x,y) = \tan\Bigl(x + u(x^2+y)\Bigr) \quad \text{with arbitrary } u(\ )\,.$$

State this qlPDE.

Problem 3 (20 marks)
For the Hamilton function

$$H = \frac{p^2}{2m} - \frac{m}{2}\gamma^2 x^2 \quad \text{with constant } m > 0 \text{ and } \gamma > 0$$

find the phase-space density $\varrho(t,x,p)$ in terms of $\varrho_0(x,p) = \varrho(t=0,x,p)$.

Problem 4 (25 marks)
What is the smallest value that you can get for

$$\int_0^1 \mathrm{d}x\,\left[\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right]^2$$

if the permissible $y(x)$ are restricted by

$$y(0) = y(1) = 0 \qquad \text{and} \qquad \int_0^1 \mathrm{d}x\,y(x) = 2\,?$$

Problem 5 (25 marks)
A point-like object [mass m, position vector $\boldsymbol{r} \mathrel{\widehat{=}} (x,y,z)$] is moving without friction on the paraboloid specified by $2z = \kappa(x^2+y^2)$ with $\kappa > 0$, while the gravitational force $m\boldsymbol{g} \mathrel{\widehat{=}} (0,0,-mg)$ is acting. Use polar coordinates in the x,y-plane, that is $(x,y) = (s\cos\varphi, s\sin\varphi)$, and find the Lagrange function $L(s,\varphi,\dot{s},\dot{\varphi})$. Then determine the corresponding Hamilton function.