Question Test 2/1

Write answers on this side of the paper only.

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$\boxed{1}$ We get successively

$AD = ABC = C, \quad AE = AAC = BC = D;$

$BA = BABB^{-1} = BB^{-1} = 1, \; BB = BAA = A,$

$BD = BBC = AC = E, \; BE = BAC = C;$

$CD = CCA = A, \; CE = CCB = B;$

$DA = CAA = CB = E, \; DB = CAB = C, \; DC = BCC = B,$

$DD = CABC = 1, \; DE = BCCB = BB = A;$

$EA = CBA = C, \quad EB = CBB = CA = D, \; EC = ACC = A,$

$ED = ACCA = AA = B, \quad EE = CBAC = 1;$

so that the completed table is

	1	A	B	C	D	E
1	1	A	B	C	D	E
A	A	B	1	E	C	D
B	B	1	A	D	E	C
C	C	D	E	1	A	B
D	D	E	C	B	1	A
E	E	C	D	A	B	1

Subgroup with three elements: $\{1, A, B\}$;

subgroup with two elements: $\underbrace{\{1, C\}; \{1, D\}; \{1, E\}}_{\text{3 all together}}$.

$\boxed{2}$ (1) Clearly, for any g_1 and g_2, the product $g_1 g_2$ is also a triplet of three real numbers.

(2) There is a unique neutral element $(0,0,0)$.

(3) There is a unique inverse to $g = (a,b,c)$, namely $g^{-1} = (-a, -b, -c+ab)$ such that $gg^{-1} = g^{-1}g = (0,0,0)$.

Question Test 2/2

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(4) Composition is associative:

$$(g_1 g_2) g_3 = (a_1+a_2+a_3,\ b_1+b_2+b_3,\ c_1+c_2+a_1 b_2+c_3+(a_1+a_2)b_3)$$

$$= (a_1+a_2+a_3,\ b_1+b_2+b_3,\ c_1+c_2+c_3+a_2 b_3+a_1(b_2+b_3))$$

$$= g_1(g_2 g_3).$$

In view of properties (1)–(4), G is a group, indeed. It is <u>not</u> Abelian because we have $(1,0,0)(0,1,0)=(1,1,1)\neq(1,1,0)=(0,1,0)(1,0,0)$, for example.

$\boxed{3}$ For

$$F(s)=\int_0^\infty dt\, e^{-st}\,\frac{\cos(at)-\cos(bt)}{t}$$

the integral in question is obtained for $F(s=0)$, and we have

$$\frac{d}{ds}F(s)=\int_0^\infty dt\, e^{-st}\left[\cos(bt)-\cos(at)\right]$$

$$=\frac{s}{s^2+b^2}-\frac{s}{s^2+a^2}=\frac{d}{ds}\ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}\ .$$

In conjunction with $F(s\to\infty)=0$, this implies

$$F(s)=\ln\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

so that

$$F(0)=\int_0^\infty dt\,\frac{\cos(at)-\cos(bt)}{t}=\ln\frac{b}{a}\ .$$

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[4] For $f(t) = \int_0^t dt'\, J_0(t-t')\, J_0(t')$ we have

$$F(s) = \left(\frac{1}{\sqrt{1+s^2}}\right)^2 \text{ because of the convolution.}$$

But $\frac{1}{1+s^2}$ is the Laplace transform of $\sin t$, so that $f(t) = \sin t$ follows,

$$\int_0^t dt'\, J_0(t-t')\, J_0(t') = \sin t.$$

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[5] Making use of the periodicity, we get

$$F(s) = \int_0^\infty dt\, e^{-st} f(t) = \int_0^\infty dt\, e^{-st} f(t+T)$$

$$= \int_T^\infty dt\, e^{-s(t-T)} f(t) = e^{sT}\left[F(s) - \int_0^T dt e^{-st} f(t)\right],$$

which we solve for $F(s)$ to arrive at

$$F(s) = \frac{1}{1-e^{-sT}} \int_0^T dt\, e^{-st} f(t), \text{ indeed.}$$

Likewise we first establish

$$G(s) = e^{sT}\left[\int_0^T dt\, e^{-st} g(t) - G(s)\right]$$

and then

$$G(s) = \frac{1}{1+e^{-sT}} \int_0^T dt\, e^{-st} g(t).$$