Question $\underline{\text{Exam}}/1$

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$\boxed{1}$ (a) (i) Generally, $\frac{\partial}{\partial t}\rho + \{\rho, H\} = 0$;

here: $\left(\frac{\partial}{\partial t} + \left(\frac{p}{m} - \gamma x\right)\frac{\partial}{\partial x} + \gamma p \frac{\partial}{\partial p}\right)\rho = 0.$

(ii) Equations for the characteristic curves are

$$dx = \left(\frac{p}{m} - \gamma x\right) dt,$$

$$dp = \gamma p \, dt.$$

Solved by

$$p(t) = p_0 e^{\gamma t},$$

$$x(t) = x_0 e^{-\gamma t} + \frac{p_0}{2\gamma m}\left(e^{\gamma t} - e^{-\gamma t}\right),$$

so that the characteristic curves are

$$p_0 = p e^{-\gamma t} = \text{const},$$

$$x_0 = x e^{\gamma t} - \frac{p}{2\gamma m}\left(e^{\gamma t} - e^{-\gamma t}\right) = \text{const}.$$

This then gives

$$\rho(t, x, p) = \rho_0\left(x e^{\gamma t} - \frac{p}{2\gamma m}\left(e^{\gamma t} - e^{-\gamma t}\right), p e^{-\gamma t}\right).$$

(iii) $\int dx\,dp\, \rho(t, x, p) = \int dx' dp' \, \rho_0(x', p')$

after substitution

$$x' = x e^{\gamma t} - \frac{p}{2\gamma m}\left(e^{\gamma t} - e^{-\gamma t}\right),$$

$$p' = p e^{\gamma t},$$

for which $dx' dp' = dx\, dp$.

Question Exam/2

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$\boxed{1}$ (b) We want $\int_0^1 dx\; x\, 2y'\, \delta y' = 0$ or

$$\underbrace{x y' \delta y \Big|_{x=0}^{1}}_{=0} - \int_0^1 dx\, \delta y \frac{d}{dx}(xy') = 0$$

$=0$ as $y'(0)=0$ and $\delta y(1)=0$,

thus we want

$$\int_0^1 dx\; \delta y\; \frac{d}{dx}(xy') = 0$$

with the constraint

$$\int_0^1 dx\; x\; \delta y = 0,$$

so that

$$\frac{d}{dx}(xy') = \lambda x$$

with Lagrange multiplier λ. This implies first

$$y' = \frac{1}{2}\lambda x,$$

then

$$y(x) = \frac{\lambda}{4}(x^2-1),$$

where $y(1)=0$ and $y'(0)=0$ are taken into account. The constraint

$$1 = \int_0^1 dx\; x\, y(x) = \frac{\lambda}{4}\left(\frac{1}{4}-\frac{1}{2}\right) = -\frac{\lambda}{16}$$

gives $\lambda = -16$ and $y(x) = 4 - 4x^2$, so that the looked-for minimal value is

$$\int_0^1 dx\; x\,(-8x)^2 = \underline{\underline{16}}.$$

Question Exam/3

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[2] (a) Neutral element: $e = (0,0,1)$.

Inverse element: $g^{-1} = (-a, -b, u^* e^{iab})$.

Composition is associative:

$$(g_1 g_2) g_3 = (a_1+a_2+a_3,\ b_1+b_2+b_3,\ u_1 u_2 e^{ia_1 b_2} u_3 e^{i(a_1+a_2)b_3}),$$

$$g_1 (g_2 g_3) = (a_1+a_2+a_3,\ b_1+b_2+b_3,\ u_1 u_2 u_3 e^{ia_2 b_3} e^{ia_1(b_2+b_3)})$$

are clearly the same: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.

(b) Cyclic subgroup needs elements $g_1, g_2, \dots, g_N = e$, such that

$g_n = g_1^n$. But $g_1^N = e = (0,0,1)$ is only possible for $g_1 = (0,0,u_1)$ with $u_1^N = 1$. Therefore, we have

$$g_n = (0, 0, e^{i\frac{2\pi}{N}n})$$

for the choice $g_1 = (0,0,e^{i\frac{2\pi}{N}})$. Other choices are possible, but they just amount to a permutation of the g_n and do not give other subgroups.

(c) Need $e^{ia_1 b_2} = e^{ia_2 b_1}$, that is: $a_1 b_2 - a_2 b_1$ must be an integer multiple of 2π.

(d) According to (c), we have $b_0 = \frac{2\pi}{a_0}$, then

$$g_{jk} = a^j b^k = (j a_0, k b_0, 1)$$

with $j, k = 0, \pm 1, \pm 2, \pm 3, \dots$, must all be in the subgroup, and we do not need any other element of G, so that these g_{jk} make up that Abelian subgroup.

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$\boxed{3}$ (a)(i) Recall $F(s) = \int_0^\infty dt\, e^{-st} f(t)$,

$$-\frac{d}{ds}F(s) = \int_0^\infty dt\, e^{-st}\, t f(s),$$

$$-f(0) + sF(s) = \int_0^\infty dt\, e^{-st} \frac{d}{dt} f(t),$$

then

$$\int_0^\infty dt\, e^{-st}\, t \frac{d^2}{dt^2} f(t) = f(0) - \frac{d}{ds}(s^2 F(s))$$

$$= f(0) - 2s F(s) - s^2 \frac{dF}{ds}.$$

Therefore,

$$\left(f(0) - 2sF(s) - s^2 \frac{dF}{ds}\right) + 2(-f(0) + sF(s)) - \frac{dF}{ds} = G(s),$$

or $$-\frac{dF}{ds} = \frac{f(0) + G(s)}{1+s^2}.$$

(ii) This implies

$$t f(t) = f(0) \sin t + \int_0^t dt' G(t') \underbrace{\sin(t-t')}_{= \sin t \cos t' - \cos t \sin t'}$$

so that

$$f(t) = \frac{\sin t}{t}\left(f(0) + \int_0^t dt' G(t') \cos t'\right) - \frac{\cos t}{t} \int_0^t dt' G(t') \sin t'.$$

[Check: The differential equation can be written as $\left(\frac{d^2}{dt^2}+1\right)(t f(t)) = g(t)$, so that $\frac{\sin t}{t}$ and $\frac{\cos t}{t}$ are obviously the homogeneous solutions.]

Question Exam/5

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$\boxed{3}$ (b)(i) $f(t) = \frac{\sin t}{t}\sin t - \frac{\cos t}{t}(1-\cos t) = \frac{1-\cos t}{t}$.

(ii) $f(t) = \frac{\sin t}{t}\left[-2+2\left(1-(\cos t)^3\right)\right] - \frac{\cos t}{t}\, 2(\sin t)^3$

$$= - \frac{\sin(2t)}{t}.$$

(c) Use $\frac{1}{t} = \int_0^\infty ds\, e^{-st}$, to get

$$\int_0^\infty dt\, \frac{J_0(at) - J_0(bt)}{t}$$

$$= \int_0^\infty ds \int_0^\infty dt\, e^{-st}\left[J_0(at) - J_0(bt)\right]$$

$$= \int_0^\infty ds \left(\frac{1}{\sqrt{s^2+a^2}} - \frac{1}{\sqrt{s^2+b^2}}\right)$$

$$= \left[\operatorname{Arsinh}(s/a) - \operatorname{Arsinh}(s/b)\right]\Big|_{s=0}^{\infty}$$

$$= \lim_{s\to\infty} \ln \frac{s+\sqrt{s^2+a^2}}{s+\sqrt{s^2+b^2}}\,\frac{b}{a} = \underline{\underline{\ln\frac{b}{a}}}.$$

Question Exam/6

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$\boxed{4}$ (a) We have

$$f(z) = \sum_{k=0}^{\infty} \Big[\underbrace{\frac{1}{3}\left(\frac{z}{3}\right)^k}_{n \geq 0} + \underbrace{\frac{3}{z^2}\left(\frac{-1}{z^2}\right)^k}_{n<0,\text{ even}} + \underbrace{\frac{1}{z}\left(\frac{-1}{z^2}\right)^k}_{n<0,\text{ odd}} \Big]$$

$$= \frac{1}{3}\,\frac{1}{1-z/3} + \left(\frac{3}{z^2}+\frac{1}{z}\right)\frac{1}{1+1/z^2}$$

$$= \frac{1}{3-z} + \frac{3+z}{1+z^2} = \frac{10}{(3-z)(1+z^2)} .$$

Singularities are simple poles at $z=3$ and $z=\pm i$, with the residues

at $z=3$, residue $=-1$;

at $z=+i$, residue $= \frac{3+i}{2i} = \frac{1}{2} - \frac{3}{2}i$;

at $z=-i$, residue $= \frac{3-i}{-2i} = \frac{1}{2} + \frac{3}{2}i$.

(b) $$f(z) = \sum_{k=0}^{\infty}\left[\frac{1}{3}\left(\frac{z}{3}\right)^k + (3+z)(-z^2)^k\right]$$

$$= \sum_{n=0}^{\infty} b_n z^n , \text{ for } |z|<1,$$

with $b_n = \left(\frac{1}{3}\right)^{n+1} + 3(-1)^{n/2}$ for n even

and $b_n = \left(\frac{1}{3}\right)^{n+1} + (-1)^{(n-1)/2}$ for n odd.

(c) This path encircles the pole at $z=3$ counterclockwise, so that the integral has the value

$$2\pi i \times (-1) = \underline{\underline{-2\pi i}} .$$

(d) This path encircles the pole at $z=+i$

Question Exam/7

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4 (d) (continued) counterclockwise and the pole at $z = -i$ clockwise, so that the integral has the value

$$2\pi i \times \left(\frac{1}{2} - \frac{3i}{2}\right) - 2\pi i \times \left(\frac{1}{2} + \frac{3i}{2}\right)$$

$$= \underline{\underline{6\pi}}.$$