

Write answers on this side of the paper only.

Do not write on either margin

$$\square (a)(i) \text{ Generally, } \frac{\partial}{\partial t} \rho + \{ \rho, H \} = 0;$$

$$\text{here: } \left(\frac{\partial}{\partial t} + \left(\frac{P}{m} - \gamma x \right) \frac{\partial}{\partial x} + \gamma p \frac{\partial}{\partial p} \right) \rho = 0.$$

(ii) Equations for the characteristic curves are

$$dx = \left(\frac{P}{m} - \gamma x \right) dt,$$

$$dp = \gamma p dt.$$

Solved by

$$p(t) = p_0 e^{\gamma t}$$

$$x(t) = x_0 e^{-\gamma t} + \frac{p_0}{2\gamma m} (e^{\gamma t} - e^{-\gamma t}),$$

so that the characteristic curves are

$$p_0 = p e^{-\gamma t} = \text{const},$$

$$x_0 = x e^{\gamma t} - \frac{p}{2\gamma m} (e^{\gamma t} - e^{-\gamma t}) = \text{const}.$$

This then gives

$$\rho(t, x, p) = \rho_0 \left(x e^{\gamma t} - \frac{p}{2\gamma m} (e^{\gamma t} - e^{-\gamma t}), p e^{-\gamma t} \right).$$

$$(iii) \int dx dp \rho(t, x, p) = \int dx' dp' \rho_0(x', p')$$

after substitution

$$x' = x e^{\gamma t} - \frac{p}{2\gamma m} (e^{\gamma t} - e^{-\gamma t}),$$

$$p' = p e^{\gamma t},$$

for which $dx' dp' = dx dp$.

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□ (b) We want $\int_0^1 dx \ x^2 y' \delta y' = 0$ or

$$x y' \delta y \Big|_{x=0}^1 - \int_0^1 dx \delta y \frac{d}{dx} (x y') = 0$$

$\underbrace{\hspace{10em}}_{=0}$ as $y'(0) = 0$ and $\delta y(1) = 0$,
thus we want

$$\int_0^1 dx \delta y \frac{d}{dx} (x y') = 0$$

with the constraint

$$\int_0^1 dx \ x \delta y = 0,$$

so that

$$\frac{d}{dx} (x y') = \lambda x$$

with Lagrange multiplier λ . This
implies first

$$y' = \frac{1}{2} \lambda x,$$

then

$$y(x) = \frac{\lambda}{4} (x^2 - 1),$$

where $y(1) = 0$ and $y'(0) = 0$ are taken
into account. The constraint

$$1 = \int_0^1 dx \ x y(x) = \frac{\lambda}{4} \left(\frac{1}{4} - \frac{1}{2} \right) = -\frac{\lambda}{16}$$

gives $\lambda = -16$ and $y(x) = 4 - 4x^2$,
so that the looked-for minimal value

is

$$\int_0^1 dx \ x (-8x)^2 = \underline{\underline{16}}.$$

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[2] (a) Neutral element: $e = (0, 0, 1)$.Inverse element: $g^{-1} = (-a, -b, u^* e^{iat})$.

Composition is associative:

$$(g_1 g_2) g_3 = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, u_1 u_2 e^{ia_1 b_2} u_3 e^{i(a_1 + a_2) b_3})$$

$$g_1 (g_2 g_3) = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, u_1 u_2 u_3 e^{ia_2 b_3} e^{ia_1 (b_2 + b_3)})$$

are clearly the same: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.

(b) Cyclic subgroup needs elements

 $g_1, g_2, \dots, g_N = e$, such that

$g_n = g_1^n$. But $g_1^N = e = (0, 0, 1)$ is only possible for $g_1 = (0, 0, u_1)$ with $u_1^N = 1$. Therefore, we have

$$g_n = (0, 0, e^{i \frac{2\pi}{N} n})$$

for the choice $g_1 = (0, 0, e^{i \frac{2\pi}{N}})$. Other choices are possible, but they just amount to a permutation of the g_n and do not give other subgroups.

(c) Need $e^{ia_1 b_2} = e^{ia_2 b_1}$, that is: $a_1 b_2 - a_2 b_1$ must be an integer multiple of 2π .

(d) According to (c), we have $b_0 = \frac{2\pi}{a_0}$, then

$$g_{jk} = a^j b^k = (j a_0, k b_0, 1)$$

with $j, k = 0, \pm 1, \pm 2, \pm 3, \dots$, must all be in the subgroup, and we do not need any other element of G , so that these g_{jk} make up that Abelian subgroup.

