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[1] (a) $\mathrm{tr}\{F^\dagger G\} = \sum_{k,l}\sum_{k',l'} f_{kl}^* g_{k'l'} \underbrace{\mathrm{tr}\{V^{-l}U^{-k}U^{k'}V^{l'}\}}$

$= \mathrm{tr}\{U^{k'-k}V^{l'-l}\}$

$= N\,\delta_{kk'}\,\delta_{ll'}$

$= \sum_{k,l}\sum_{k',l'} f_{kl}^* g_{k'l'}\, N\delta_{kk'}\,\delta_{ll'}$

$= N\sum_{k,l} f_{kl}^* g_{kl}$.

(b) $\sum_{k=1}^{N} \langle u_k| = \sum_{k,l=1}^{N} \langle u_k|v_l\rangle\langle v_l|$

$= \frac{1}{\sqrt{N}}\sum_{k,l=1}^{N} e^{i\frac{2\pi}{N}kl}\langle v_l|$

$= \frac{1}{\sqrt{N}}\sum_{l=1}^{N} N\delta_{lN} = \sqrt{N}\,\langle v_N|$.

Likewise, $\sum_{l=1}^{N} |v_l\rangle = |u_N\rangle\sqrt{N}$.

[2] (a) Characteristically $|\langle q|\gamma\rangle|^2 = \frac{1}{2\pi}$ does not depend on the quantum numbers q and γ, so that Q, Γ are a complementary pair.

(b) $\int_0^\infty \frac{dq}{q}\langle\gamma|q\rangle\langle q|\gamma'\rangle = \int_0^\infty \frac{dq}{q}\, q^{-i(\gamma-\gamma')/2\pi}$

$q = e^x$ $\;\stackrel{\downarrow}{=} \int_{-\infty}^{\infty} dx\,\frac{1}{2\pi}\,e^{-ix(\gamma-\gamma')} = \delta(\gamma-\gamma')$,

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which states that

$$\delta(\gamma-\gamma') = \langle\gamma|\gamma'\rangle = \langle\gamma|\left(\int_0^\infty \frac{dq}{q} |q\rangle\langle q|\right)|\gamma'\rangle$$

and implies $\int_0^\infty \frac{dq}{q} |q\rangle\langle q| = 1.$

$$\int_{-\infty}^{\infty} d\gamma \; \langle q|\gamma\rangle\langle\gamma|q'\rangle = \int_{-\infty}^{\infty} d\gamma \; (q/q')^{i\gamma}/2\pi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\gamma \; e^{i\gamma \log(q/q')} = \delta(\log(q/q'))$$

$$= q\,\delta(q-q')$$

$$= \langle q|q'\rangle,$$

and $\int_{-\infty}^{\infty} d\gamma \; |\gamma\rangle\langle\gamma| = 1$ follows.

(c) Consider $\langle q|Q^{i\gamma'}|\gamma\rangle = q^{i\gamma'}\langle q|\gamma\rangle = \frac{q^{i(\gamma+\gamma')}}{\sqrt{2\pi}}$

$$= \langle q|\gamma+\gamma'\rangle,$$

and the completeness of the bras $\langle q|$ implies

$$Q^{i\gamma'}|\gamma\rangle = |\gamma+\gamma'\rangle.$$

(d) Compare $e^{i\beta\Gamma}Q^{i\gamma}|\gamma'\rangle = e^{i\beta\Gamma}|\gamma+\gamma'\rangle = |\gamma+\gamma'\rangle e^{i\beta(\gamma+\gamma')}$

with $Q^{i\gamma}e^{i\beta\Gamma}|\gamma'\rangle = Q^{i\gamma}|\gamma'\rangle e^{i\beta\gamma'} = |\gamma+\gamma'\rangle e^{i\beta\gamma'}$

and conclude that $e^{i\beta\Gamma}Q^{i\gamma} = e^{i\beta\gamma}Q^{i\gamma}e^{i\beta\Gamma}$.

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$\boxed{3}$ (a) For example, consider an arbitrary power of A:

$$e^{-\epsilon B} A^n e^{\epsilon B} = \left(e^{-\epsilon B} A e^{\epsilon B}\right)^n$$

by inspection, then

$$e^{-\epsilon B} e^{A} e^{\epsilon B} = e^{e^{-\epsilon B} A e^{\epsilon B}}$$

follows by summation of the power series for e^A.

(b) To first order in ϵ, we have

$$e^{-\epsilon B} e^{A} e^{\epsilon B} = e^{A} + \epsilon\left[e^{A}, B\right]$$

and

$$e^{-\epsilon B} A e^{\epsilon B} = A + \underbrace{\epsilon[A,B]}_{\equiv \delta A} = A + \delta A.$$

So

$$\delta e^{A} = \epsilon\left[e^{A}, B\right] = \int_0^1 dx\, e^{(1-x)A}\, \delta A\, e^{xA}$$

$$= \int_0^1 dx\, e^{(1-x)A} \epsilon [A,B] e^{xA},$$

giving $\left[e^{A}, B\right] = \int_0^1 dx\, e^{(1-x)A} [A,B] e^{xA}$.

$\boxed{4}$ (a) For any eigenvalue $a > 0$ of A we have

$$\int_0^\infty d\alpha \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+a}\right) = \int_0^\infty d\alpha \frac{d}{d\alpha} \log \frac{\alpha+1}{\alpha+a}$$

$$= \log \frac{\alpha+1}{\alpha+a}\Bigg|_{\alpha=0}^{\infty} = \log 1 - \log \frac{1}{a} = \log a$$

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and

$$\int_0^\infty d\beta \, \frac{e^{-\beta} - e^{-\beta a}}{\beta} = \int_0^\infty d\beta \int_1^a d\alpha \, e^{-\alpha\beta}$$

$$= \int_1^a d\alpha \int_0^\infty d\beta \, e^{-\alpha\beta} = \int_1^a d\alpha \, \frac{1}{\alpha} = \log a,$$

so that the two integrals give the correct values for all eigenvalues of a, as they should.

(b) Recall $\delta \frac{1}{X} = -\frac{1}{X} \delta X \frac{1}{X}$ and use this for $X = \alpha + A$, $\delta X = \delta A$, then

$$\delta \log A = \delta \int_0^\infty d\alpha \left(\frac{1}{\alpha+1} - \frac{1}{\alpha+A} \right)$$

$$= \int_0^\infty d\alpha \, \frac{1}{\alpha+A} \, \delta A \, \frac{1}{\alpha+A} \, .$$