Question Test 2/1

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1 (a)
$$\rho(\vec r,t) = |\psi(\vec r,t)|^2$$
$$= \left[c_0^2 + (\vec c\cdot\vec r)^2 + 2c_0\,\vec c\cdot\vec r\cos(\omega t)\right] e^{-k^2 r^2};$$

$$\vec j(\vec r,t) = \frac{\hbar}{M}\,\mathrm{Im}\left(\psi(\vec r,t)^* \vec\nabla \psi(\vec r,t)\right)$$
$$= \frac{\hbar}{M}\,\mathrm{Im}\Big(\left(c_0 + \vec c\cdot\vec r e^{i\omega t}\right) e^{-\frac12 k^2 r^2}$$
$$\times\left[\left(\vec c e^{-i\omega t} - (c_0 + \vec c\cdot\vec r e^{-i\omega t}) k^2 \vec r\right) e^{-\frac12 k^2 r^2}\right]\Big)$$

$$= \frac{\hbar}{M}\,\mathrm{Im}\left(c_0 \vec c\, e^{-i\omega t} e^{-k^2 r^2}\right)$$
$$= -\frac{\hbar}{M}\, c_0 \vec c \sin(\omega t)\, e^{-k^2 r^2}.$$

(b)
$$\frac{\partial}{\partial t}\rho = -2\omega c_0\, \vec c\cdot\vec r \sin(\omega t)\, e^{-k^2 r^2},$$
$$\vec\nabla\cdot\vec j = 2\frac{\hbar k^2}{M} c_0\, \vec c\cdot\vec r \sin(\omega t)\, e^{-k^2 r^2},$$
$$\frac{\partial}{\partial t}\rho + \vec\nabla\cdot\vec j = 2\left(-\omega + \frac{\hbar k^2}{M}\right) c_0\, \vec c\cdot\vec r \sin(\omega t)\, e^{-k^2 r^2}$$
$$= 0,$$

implies $\boxed{\hbar k^2 = M\omega}$ or $\hbar\omega = \frac{(\hbar k)^2}{M}$.

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$\boxed{2}$ (a) According to the notes, these expressions are valid for $x \le 0$ and $x \ge L$, respectively, but here the force acts only at $x = L/2$, so that they continue to be valid also for $0 \le x \le L/2$ and $L/2 \le x \le L$.

(b) $\phi(k, x = \frac{L}{2} - 0) = \frac{1}{\sqrt{k}}\left(\phi_+(k,0)e^{ikL/2} + \phi_-(k,0)e^{-ikL/2}\right)$

$= \phi(k, x = \frac{L}{2} + 0) = \frac{1}{\sqrt{k}}\left(\phi_+(k,L)e^{-ikL/2} + \phi_-(k,L)e^{ikL/2}\right)$

gives

$$\boxed{\underbrace{\phi_+(k,L) - \phi_-(k,0)}_{\text{"out"}} = \underbrace{\left[\phi_+(k,0) - \phi_-(k,L)\right]}_{\text{"in"}} e^{ikL}}.$$

(c) $\phi(k,x)$ obeys the differential equation

$\left[\frac{\partial^2}{\partial x^2} + k^2 - \frac{2M}{\hbar^2}V(x)\right]\phi(k,x) = 0$, which

for the given $V(x)$ is

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right)\phi(k,x) = -\frac{2}{a}\,\delta(x - \tfrac{L}{2})\,\phi(k,x)$$
$$= -\frac{2}{a}\,\delta(x - \tfrac{L}{2})\,\phi(k, \tfrac{L}{2}).$$

The δ function on the right implies that $\frac{\partial}{\partial x}\phi$ has a discontinuous "jump" of $-\frac{2}{a}\phi(k, \frac{L}{2})$ at $x = \frac{L}{2}$, that is

$$\frac{\partial\phi}{\partial x}(k, \tfrac{L}{2}+0) - \frac{\partial\phi}{\partial x}(k, \tfrac{L}{2}-0) = -\frac{2}{a}\,\phi(k, \tfrac{L}{2}).$$

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$$LHS = i\sqrt{k}\left(\phi_+(k,L)\,e^{-ikL/2} - \phi_-(k,L)\,e^{ikL/2}\right)$$
$$\quad - i\sqrt{k}\left(\phi_+(k,0)\,e^{ikL/2} - \phi_-(k,0)\,e^{-ikL/2}\right)$$

$$= i\sqrt{k}\Big[\left(\phi_+(k,L) + \phi_-(k,0)\right)e^{-ikL/2}$$
$$\quad - \left(\phi_+(k,0) + \phi_-(k,L)\right)e^{ikL/2}\Big],$$

$$RHS = -\frac{1}{a}\,\phi\!\left(k, x=\tfrac{L}{2}+0\right) - \frac{1}{a}\,\phi\!\left(k, x=\tfrac{L}{2}-0\right)$$

$$= -\frac{1}{a\sqrt{k}}\Big[\left(\phi_+(k,L) + \phi_-(k,0)\right)e^{-ikL/2}$$
$$\quad + \left(\phi_+(k,0) + \phi_-(k,0)\right)e^{ikL/2}\Big],$$

so that $LHS = RHS$ gives

$$\boxed{(1+ika)\underbrace{\left[\phi_+(k,L) + \phi_-(k,0)\right]}_{\text{"out"}} = -(1-ika)\underbrace{\left[\phi_+(k,0)+\phi_-(k,L)\right]}_{\text{"in"}}e^{ikL}.}$$

(d) Here $k(x) = k$ for $x < \frac{L}{2}$ and $x > \frac{L}{2}$, so that

$$\alpha = \int_0^L dx\, k(x) = kL\,, \quad e^{ikL} = e^{i\alpha},$$

and we have

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \phi_+(k,L) \\ \phi_-(k,0) \end{pmatrix} = e^{i\alpha}\begin{pmatrix} 1 & -1 \\ \mu & \mu \end{pmatrix}\begin{pmatrix} \phi_+(k,0) \\ \phi_-(k,0) \end{pmatrix}$$

with $\mu \equiv -\dfrac{1-ika}{1+ika} = \dfrac{ka+i}{ka-i}$ for the two relations from (b) and (c)

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Multiply by $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ to solve for the "out" components, which gives

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ \mu & \mu \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mu+1 & \mu-1 \\ \mu-1 & \mu+1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ka}{ka-i} & \frac{i}{ka-i} \\ \frac{i}{ka-i} & \frac{ka}{ka-i} \end{pmatrix}.$$

$\boxed{3}$ (a) We have $\text{prob}(g \to g, t) = 1 - \left(\frac{\delta E}{\hbar} t\right)^2$ with

$$\delta E = \sqrt{\langle g|H_1^2|g\rangle - \langle g|H_1|g\rangle^2}, \text{ where}$$

$H_1|g\rangle = |e\rangle \hbar\Omega$, so that $\langle g|H_1|g\rangle = 0$ and $\langle g|H_1^2|g\rangle = (\hbar\Omega)^2$. It follows that $\delta E = \hbar\Omega$ and $\gamma = \delta E/\hbar = \Omega$.

(b) Since $\sigma = |g\rangle\langle e|$, $\sigma^+ = |e\rangle\langle g|$, $\sigma^+\sigma = |e\rangle\langle e|$, we have

$\sigma(\sigma^+\sigma) = \sigma$ so that $\sigma f(\sigma^+\sigma) = \sigma f(1)$,

$(\sigma^+\sigma)\sigma^+ = \sigma^+$ so that $f(\sigma^+\sigma)\sigma^+ = f(1)\sigma^+$,

$\sigma^+(\sigma^+\sigma) = 0$ so that $\sigma^+ f(\sigma^+\sigma) = \sigma^+ f(0)$,

and $(\sigma^+\sigma)\sigma = 0$ so that $f(\sigma^+\sigma)\sigma = f(0)\sigma$.

Therefore

$$\overline{H}_1(t) = e^{i\omega t\sigma^+\sigma}\,\hbar\Omega(\sigma^+ + \sigma)\,e^{-i\omega t\sigma^+\sigma}$$

$$= \hbar\Omega\left(e^{i\omega t}\sigma^+ + \sigma e^{-i\omega t}\right).$$

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(c) The probability amplitude for $g \to e$ to first order in Ω is

$$-\frac{i}{\hbar}\int_0^T dt\, \langle e|\, \overline{H}_1(t)\, |g\rangle$$

$$= -i\Omega \int_0^T dt \left(\underbrace{\langle e|\sigma^+|g\rangle}_{=1} e^{i\omega t} + \underbrace{\langle e|\sigma|g\rangle}_{=0} e^{-i\omega t} \right)$$

$$= -i\Omega\, \frac{e^{i\omega T}-1}{i\omega} = -\frac{\Omega}{\omega}\, e^{i\omega T/2} \underbrace{\left(e^{i\omega T/2} - e^{-i\omega T/2}\right)}_{2i \sin\frac{\omega T}{2}},$$

so that the probability is

$$\mathrm{prob}(g \to e, T) = 4\,\frac{\Omega^2}{\omega^2}\left(\sin\frac{\omega T}{2}\right)^2$$

$$= (\Omega T)^2 \left(\frac{\sin\frac{\omega T}{2}}{\omega T/2}\right)^2 .$$

$\boxed{4}$ (a) $$f_+(\vec{k}',\vec{k}) = -\frac{M}{2\pi\hbar^2}\int (d\vec{r})\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, V(\vec{r}-\vec{a})$$

$$= -\frac{M}{2\pi\hbar^2}\int (d\vec{r})\, e^{i(\vec{k}-\vec{k}')\cdot(\vec{r}+\vec{a})}\, V(\vec{r})$$

$$= e^{i(\vec{k}-\vec{k}')\cdot\vec{a}}\, f(\vec{k}',\vec{k})$$

so that $\frac{d\sigma_+}{d\Omega} = |f_+(\vec{k}',\vec{k})|^2 = |f(\vec{k}',\vec{k})|^2 = \frac{d\sigma}{d\Omega}$.

(b) likewise $f_-(\vec{k}',\vec{k}) = e^{-i(\vec{k}-\vec{k}')\cdot\vec{a}}\, f(\vec{k}',\vec{k})$

and $\frac{d\sigma_-}{d\Omega} = \frac{d\sigma}{d\Omega}$.

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(c) Here $f_2(\vec{k}',\vec{k}) = f_+(\vec{k}',\vec{k}) + f_-(\vec{k}',\vec{k})$

$$= \left(e^{i(\vec{k}-\vec{k}')\cdot\vec{a}} + e^{-i(\vec{k}-\vec{k}')\cdot\vec{a}}\right) f(\vec{k}',\vec{k})$$

$$= 2\cos\left((\vec{k}-\vec{k}')\cdot\vec{a}\right) f(\vec{k}',\vec{k})$$

and therefore

$$\frac{d\sigma_2}{d\Omega} = |f_2(\vec{k}',\vec{k})|^2$$

$$= 4\left(\cos\left((\vec{k}-\vec{k}')\cdot\vec{a}\right)\right)^2 \frac{d\sigma}{d\Omega}.$$

We get strong enhancement (factor 4) for $\vec{k}'$ such that $(\vec{k}-\vec{k}')\cdot\vec{a} = 0, \pm\pi, \pm 2\pi, \ldots$ (constructive interference) and complete cancelation for $\vec{k}'$ such that $(\vec{k}-\vec{k}')\cdot\vec{a} = \pm\pi/2, \pm 3\pi/2, \ldots$ (destructive interference).