Problem 1 (20=10+10 points)
The state of a particle of mass M is described by the wave function

$$\psi(\vec{r},t) = \left(c_0 + \vec{c}\cdot\vec{r}\,\mathrm{e}^{-\mathrm{i}\omega t}\right)\mathrm{e}^{-\frac{1}{2}\kappa^2 r^2}\,,$$

where c_0 and $\vec{c}$ as well as κ and ω are real parameters.

(a) Find the probability density $\rho(\vec{r},t)$
and the probability current density $\vec{j}(\vec{r},t)$.

(b) Which relation between κ and ω follows from the continuity equation obeyed by $\rho(\vec{r},t)$ and $\vec{j}(\vec{r},t)$?

Problem 2 (30=6+6+10+8 points)
Consider scattering in one dimension (see pages 101–106 of the notes) by the delta potential

$$V(x) = -\frac{\hbar^2}{Ma}\delta(x - L/2)\,,$$

where a is a length parameter with $a > 0$ (attractive potential) or $a < 0$ (repulsive potential).

(a) Explain why $\phi(k,x)$ is of the form

$$\phi(k,x) = \begin{cases} \dfrac{1}{\sqrt{k}}\left[\phi_+(k,0)\,\mathrm{e}^{\mathrm{i}kx} + \phi_-(k,0)\,\mathrm{e}^{-\mathrm{i}kx}\right] & \text{for } x \le L/2\,, \\ \dfrac{1}{\sqrt{k}}\left[\phi_+(k,L)\,\mathrm{e}^{\mathrm{i}k(x-L)} + \phi_-(k,L)\,\mathrm{e}^{-\mathrm{i}k(x-L)}\right] & \text{for } x \ge L/2\,. \end{cases}$$

(b) Which relation among the "in" amplitudes $\phi_+(k,0)$, $\phi_-(k,L)$ and the "out" amplitudes $\phi_+(k,L)$, $\phi_-(k,0)$ follows from the continuity of $\phi(k,x)$ at $x = L/2$?

(c) Why is the derivative of $\phi(k,x)$ discontinuous at $x = L/2$ as stated by

$$\left.\frac{\partial\phi(k,x)}{\partial x}\right|_{x=L/2-0}^{x=L/2+0} = -\frac{2}{a}\phi(k, x = L/2)\,?$$

Use this to find a second relation among the "in" and "out" amplitudes.

(d) Now establish $\alpha = kL$ for the phase on pages 105/106, and then get the scattering matrix $S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}$ from the relations found in parts (b) and (c). Express the matrix elements of S in terms of k and a.

Problem 3 (25=10+9+6 points)
A two-level atom with unperturbed Hamilton operator $H_0 = \hbar\omega\sigma^\dagger\sigma$ (see page 65 of the notes) is exposed to a time-independent perturbation that is specified by

$$H_1 = \hbar\Omega(\sigma^\dagger + \sigma) \qquad \text{with} \quad \Omega > 0\,.$$

At the initial time, the atom is in the ground state $|\mathrm{g}\rangle$ of H_0.

(a) For short times, the probability $\mathrm{prob}(\mathrm{g} \to \mathrm{g}, t)$ for remaining in the ground state of H_0 is of the form $\mathrm{prob}(\mathrm{g} \to \mathrm{g}, t) = 1 - (\gamma t)^2$. Determine the value of γ.

(b) Express $\overline{H_1}(t) = \mathrm{e}^{\mathrm{i}H_0 t/\hbar} H_1 \mathrm{e}^{-\mathrm{i}H_0 t/\hbar}$ as a linear combination of σ, $\sigma^\dagger$, $\sigma^\dagger\sigma$, and $\sigma\sigma^\dagger$.

(c) What is the probability, to lowest order in Ω, for finding the atom in the excited state $|\mathrm{e}\rangle$ of H_0 after time T has elapsed?

Problem 4 (25=12+3+10 points)
In the Born approximation (see page 125 of the notes), a certain scattering potential $V(\vec{r})$, which is centered at $\vec{r} = 0$, has the scattering amplitude $f\left(\vec{k}', \vec{k}\right)$ and the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

(a) What are the scattering amplitude $f_+\left(\vec{k}', \vec{k}\right)$ and differential cross section $\frac{\mathrm{d}\sigma_+}{\mathrm{d}\Omega}$ for the potential $V_+(\vec{r}) = V(\vec{r} - \vec{a})$, centered at $\vec{r} = \vec{a}$?

(b) What are the corresponding $f_-\left(\vec{k}', \vec{k}\right)$ and $\frac{\mathrm{d}\sigma_-}{\mathrm{d}\Omega}$ for $V_-(\vec{r}) = V(\vec{r} + \vec{a})$, centered at $\vec{r} = -\vec{a}$?

(c) Now determine the differential cross section $\frac{\mathrm{d}\sigma_2}{\mathrm{d}\Omega}$ for the two-center scattering potential $V_2(\vec{r}) = V_+(\vec{r}) + V_-(\vec{r})$.