Question Exam/1

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1 (a) From Test 2/Problem 2 we know that these relations hold for the two single delta potentials, with

$$\begin{pmatrix} t & r \\ r & t \end{pmatrix} = e^{i\alpha} \begin{pmatrix} \frac{ka}{ka-i} & \frac{i}{ka-i} \\ \frac{i}{ka-i} & \frac{ka}{ka-i} \end{pmatrix} \text{ where } \alpha = kL.$$

Writing $ka = \cot\beta$, we have therefore

$$t = e^{i\alpha} \frac{\cot\beta}{\cot\beta - i} = e^{i\alpha} \frac{\cos\beta}{\cos\beta - i\sin\beta} = e^{i(\alpha+\beta)}\cos\beta$$

and

$$r = \frac{i}{ka} t = i t \tan\beta = i e^{i(\alpha+\beta)} \sin\beta .$$

(b) We have the two equations

$$\begin{pmatrix} \phi_+(L) \\ \phi_-(-L) \end{pmatrix} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \begin{pmatrix} \phi_+(-L) \\ \phi_-(L) \end{pmatrix} + t \begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix}$$

where we need $\phi_\pm(0)$, which we get from the other two equations, namely

$$\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix} = t \begin{pmatrix} \phi_+(-L) \\ \phi_-(L) \end{pmatrix} .$$

Accordingly,

$$\begin{pmatrix} T & R \\ R & T \end{pmatrix} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} + t^2 \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} + \frac{t^2}{1-r^2} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} ,$$

so that $\quad T = \dfrac{t^2}{1-r^2}$

and $\quad R = \dfrac{r}{1-r^2} (1 - r^2 + t^2) .$

Question Exam/2

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(c) We need $0 = 1 - r^2 + t^2 = 1 + e^{2i(\alpha+\beta)}$

or

$$\cos(\alpha+\beta) = 0,$$

or $\cos\alpha \cos\beta = \sin\alpha \sin\beta$,

so that

$$\tan\alpha = \cot\beta$$

or, finally $\boxed{\tan(kL) = ka}$.

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[2] (a) For the Yukawa potential we have

$$-\frac{M}{2\pi\hbar^2}\int (d\vec{r})\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, \frac{V_0}{\kappa r} e^{-\kappa r}$$

$$= -\frac{2MV_0/\kappa}{\hbar^2(\kappa^2+q^2)} \equiv f_0(\theta) \text{ with } q = 2k\sin\frac{\theta}{2}.$$

Therefore,

$$f(\theta) = 2\cos\left((\vec{k}-\vec{k}')\cdot\vec{a}\right) f_0(\theta)$$

[see Test 2/Problem 4] where $\vec{k}\cdot\vec{a} = ka$ and $\vec{k}'\cdot\vec{a} = \vec{k}'\cdot\vec{k}\, a/k = ka\cos\theta$, so that

$$f(\theta) = 2\cos\left(ka(1-\cos\theta)\right) f_0(\theta)$$

$$= 2\cos\left(2ka\left(\sin\frac{\theta}{2}\right)^2\right) f_0(\theta),$$

and $\frac{d\sigma}{d\Omega} = 4\left(\cos\left(2ka\left(\sin\frac{\theta}{2}\right)^2\right)\right)^2 |f_0(\theta)|^2$.

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(b) For $\frac{d\sigma}{d\Omega} = 0$ we need $\cos(ka(1-\cos\theta)) = 0$, so that

$$\cos\theta = \tfrac{2}{3}: \quad \cos\left(\tfrac{1}{3}ka\right) = 0,$$

$$\cos\theta = 0: \quad \cos(ka) = 0,$$

$$\cos\theta = -\tfrac{2}{3}: \quad \cos\left(\tfrac{5}{3}ka\right) = 0,$$

which tell us that $ka = \frac{3\pi}{2} = 2\pi \times \frac{3}{4}$. It follows that

$$\boxed{a = \tfrac{3}{4}\lambda}.$$

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3 (a) We have

$$(A^+\sigma + \sigma^+ A)^2 = A^+A\underbrace{\sigma\sigma^+}_{=1-\sigma^+\sigma} + \underbrace{AA^+}_{=A^+A+1}\sigma^+\sigma = A^+A + \sigma^+\sigma,$$

since $\sigma^2 = 0$ and $\sigma^{+2} = 0$. This tells us that, essentially, H_0 is the square of H_1, so that the two parts of H commute, implying

$$[H(t_1), H(t_2)] = 0$$

for all t_1 and t_2.

Question Exam/4

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(b) $\alpha(t) = \langle e,0,t|\rangle$, $\beta(t) = \langle g,1,t|0\rangle$ give

$$i\hbar \frac{\partial}{\partial t}\alpha(t) = \langle e,0,t|H|\rangle = \hbar\omega\langle e,0,t|\rangle - \hbar\Omega\langle g,1,t|\rangle$$

$$= \hbar\omega\,\alpha(t) - \hbar\Omega(t)\beta(t),$$

$$i\hbar\frac{\partial}{\partial t}\beta(t) = \langle g,1,t|H|\rangle = \hbar\omega\langle g,1,t|\rangle - \hbar\omega\langle e,0,t|\rangle$$

$$= \hbar\omega\beta(t) - \hbar\Omega(t)\alpha(t),$$

or

$$\frac{\partial}{\partial t}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = i\begin{pmatrix}-\omega & \Omega(t)\\ \Omega(t) & -\omega\end{pmatrix}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}.$$

(c) Since the eigenvectors of $\uparrow$ are $(1,1)$ and $(1,-1)$, the time dependence of $\alpha\pm\beta$ is particularly simple:

$$\frac{\partial}{\partial t}(\alpha(t)\pm\beta(t)) = i(-\omega\pm\Omega(t))(\alpha(t)\pm\beta(t)),$$

with the consequence

$$\alpha(t)\pm\beta(t) = (\alpha(0)\pm\beta(0))\,e^{-i\omega t \pm i\int_0^t dt'\,\Omega(t')}.$$

For $\alpha(0)=1$, $\beta(0)=0$ and $t=T$, this gives

$$\alpha(T)\pm\beta(T) = e^{-i\omega T}\,e^{\pm i\int_0^T dt\,\Omega(t)},$$

or with

$$\int_0^T dt\,\Omega(t) = 2\int_0^{T/2} dt\,\frac{2\pi t}{T^2} = \frac{\pi}{2},$$

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$$\alpha(T) \pm \beta(T) = e^{-i\omega T} e^{\pm i\frac{\pi}{2}} = \pm i\, e^{-i\omega T}.$$

Thus, $\alpha(T) = 0$, $\beta(T) = i\, e^{-i\omega T}$ and the probability in question is

$$|\beta(T)|^2 = 1.$$

$\boxed{4}$ (a) One electron: $|1\uparrow\rangle$, the other: $|2\downarrow\rangle$, together

$$|\,\rangle = \big(|\underbrace{1\uparrow}_{\text{electron 1}}, \underbrace{2\downarrow}_{\text{electron 2}}\rangle - |2\downarrow, 1\uparrow\rangle\big)/\sqrt{2}.$$

Singlet component: $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, has spatial wave fuction

$$\psi_s(\vec{r}_1, \vec{r}_2) = \langle \vec{r}_1, \vec{r}_2 | \big(|12\rangle + |21\rangle\big)/\sqrt{2}^{\,2}$$

$$= \frac{1}{2}\left[\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) + \psi_2(\vec{r}_1)\psi_1(\vec{r}_2)\right];$$

triplet components: $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, has spatial wave function

$$\psi_t(\vec{r}_1, \vec{r}_2) = \langle \vec{r}_1, \vec{r}_2 | \big(|12\rangle - |21\rangle\big)/\sqrt{2}^{\,2}$$

$$= \frac{1}{2}\left[\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) - \psi_2(\vec{r}_1)\psi_1(\vec{r}_2)\right].$$

(b) These probabilities are

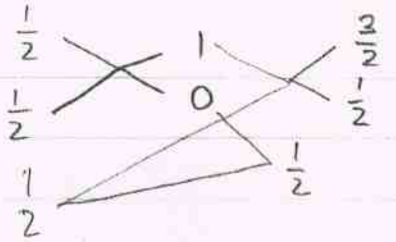
$$\int (d\vec{r}_1)(d\vec{r}_2)\, |\psi_s(\vec{r}_1, \vec{r}_2)|^2 = \frac{1}{4}\big(1 + 1 + \langle 1|2\rangle\langle 2|1\rangle + \langle 2|1\rangle\langle 1|2\rangle$$

$$= \frac{1}{2} + \frac{1}{2}\,|\langle 1|2\rangle|^2 = \frac{1+\gamma}{2}$$

and $\int (d\vec{r}_1')(d\vec{r}_2') \, |\psi_t(\vec{r}_1, \vec{r}_2)|^2 = \frac{1-\gamma}{2}$,

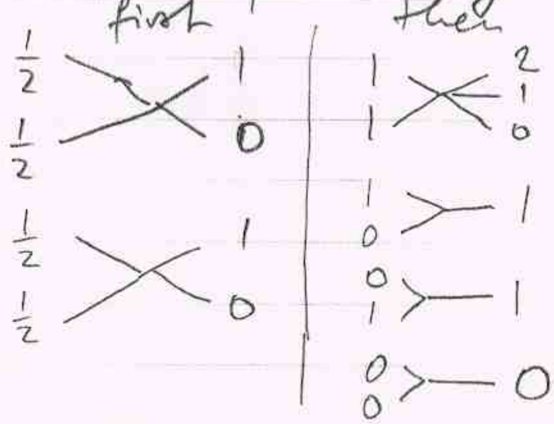
respectively.

(c) The three angular momenta of $\frac{1}{2}$ each can be coupled in this manner



We begin with $2^3 = 8$ states which are grouped into 4 states with $s = \frac{3}{2}$ and 2 times 2 states with $s = \frac{1}{2}$, together $4+2+2 = 8$ states.

(d) Here one possibility is



so that the $2^4 = 16$ states are grouped into 5 states with $s=2$, 3 times 3 states with $s=1$, and 2 times 1 state with $s=0$, together $5 + 3\times 3 + 2\times 1 = 16$ states.