1. Scattering in one dimension. A particle of mass M moves along the x-axis with energy $E = \dfrac{(\hbar k)^2}{2M}$ $(k > 0)$ and is scattered by the double delta potential

$$V(x) = -\frac{\hbar^2}{Ma}\delta(x - L/2) - \frac{\hbar^2}{Ma}\delta(x + L/2)\,.$$

The length parameter a determines the strength of the potential, with $a > 0$ for an attractive potential and $a < 0$ for a repulsive potential. As usual, denote the wave function for the given k value by $\phi(x)$, and decompose $\phi(x)$ into the right-moving part $\phi_+(x)$ and the left-moving part $\phi_-(x)$.

(a) Show that the action of the individual delta potentials can be summarized by

$$\begin{pmatrix}\phi_+(0)\\ \phi_-(-L)\end{pmatrix} = \begin{pmatrix} t & r \\ r & t\end{pmatrix}\begin{pmatrix}\phi_+(-L)\\ \phi_-(0)\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\phi_+(L)\\ \phi_-(0)\end{pmatrix} = \begin{pmatrix} t & r \\ r & t\end{pmatrix}\begin{pmatrix}\phi_+(0)\\ \phi_-(L)\end{pmatrix}$$

with the transmission amplitude $t = \mathrm{e}^{\mathrm{i}(\alpha + \beta)}\cos\beta$ and the reflection amplitude $r = \mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha + \beta)}\sin\beta$, where $\alpha = kL$ and $\cot\beta = ka$. [10 marks]

(b) Determine the 2×2 scattering matrix for the total scattering potential $V(x)$, that is: find the transmission coefficient R and the reflection coefficient T in

$$\begin{pmatrix}\phi_+(L)\\ \phi_-(-L)\end{pmatrix} = \begin{pmatrix} T & R \\ R & T\end{pmatrix}\begin{pmatrix}\phi_+(-L)\\ \phi_-(L)\end{pmatrix}$$

in terms of t and r. [10 marks]

(c) Which relation must be obeyed by ka and kL so that the reflection probability $|R|^2$ vanishes? [5 marks]

2. Scattering in three dimensions. A particle of mass M and wave vector $\vec{k}$ is scattered by a double Yukawa potential

$$V(\vec{r}) = Y(\vec{r} - \vec{a}) + Y(\vec{r} + \vec{a}) \quad \text{with} \quad Y(\vec{r}) = \frac{V_0}{\kappa r}\mathrm{e}^{-\kappa r}$$

where $\kappa > 0$ and $V_0 \neq 0$, and $\vec{a}$ is parallel to $\vec{k}$, that is: $\vec{k}\cdot\vec{a} = ka > 0$.

(a) Find the scattering amplitude $f(\theta)$ and the differential cross section $\dfrac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta)$ in Born approximation. [12 marks]

(b) It is observed that no scattering occurs in the three directions for which the scattering angle θ is such that $\cos\theta = 0$ or $\cos\theta = \pm 2/3$. How big is the spacing a between the scattering centers in terms of the de Broglie wavelength $\lambda = 2\pi/k$? [13 marks]

3. Time-dependent interaction. A two-level atom (ground state ket $|\mathrm{g}\rangle$, excited state ket $|\mathrm{e}\rangle$, energy spacing $\hbar\omega > 0$, transition operator $\sigma = |\mathrm{g}\rangle\langle\mathrm{e}|$) is resonant with a single photon mode (harmonic-oscillator ladder operators A, $A^\dagger$), to which it couples by the time-dependent Rabi frequency $\Omega(t)$. The dynamics is governed by the Hamilton operator $H(t) = H_0 + H_1(t)$ with

$$H_0 = \hbar\omega(\sigma^\dagger\sigma + A^\dagger A) \quad \text{and} \quad H_1(t) = -\hbar\Omega(t)(A^\dagger\sigma + \sigma^\dagger A)\,,$$

where

$$\Omega(t) = \begin{cases} 2\pi t/T^2 & \text{for} \quad 0 < t < T/2\,, \\ 2\pi(T-t)/T^2 & \text{for} \quad T/2 < t < T\,, \\ 0 & \text{for} \quad t < 0 \text{ and } t > T\,. \end{cases}$$

(a) Show first that $(A^\dagger\sigma + \sigma^\dagger A)^2 = \sigma^\dagger\sigma + A^\dagger A$, and then evaluate the commutator $[H(t_1), H(t_2)]$. [8 marks]

(b) Denote by $\alpha(t)$ the probability amplitude for "atom excited and no photons at time t" and by $\beta(t)$ the probability amplitude for "atom in the ground state and one photon at time t" and state the coupled Schrödinger equations that they obey. [8 marks]

(c) Solve these Schrödinger equations to answer this question: If at time $t = 0$ the atom is excited and there is no photon, what is the probability that the atom is de-excited and one photon present at time $t = T$? [9 marks]

4. Indistinguishable particles. There are two electrons, one has spin up in the z direction and the spatial wave function $\psi_1(\vec{r}) = \langle\vec{r}|1\rangle$, the other has spin down in the z direction and the spatial wave function $\psi_2(\vec{r}) = \langle\vec{r}|2\rangle$. Hereby, $\langle 1|1\rangle = 1 = \langle 2|2\rangle$, while $\gamma = \langle 1|2\rangle$ is arbitrary.

(a) Determine the spatial two-electron wave functions $\psi_\mathrm{s}(\vec{r}_1, \vec{r}_2)$ and $\psi_\mathrm{t}(\vec{r}_1, \vec{r}_2)$ for the singlet and triplet components, respectively. [9 marks]

(b) State the probabilities for finding the electron pair in the singlet and triplet sector. Express your answers in terms of γ. [6 marks]

(c) Consider all possible spin states of three electrons. How many spin states are there all together? How many of them belong to total spin $\frac{3}{2}$, how many to total spin $\frac{1}{2}$? [6 marks]

(d) What is the corresponding situation for the spin states of four electrons? [4 marks]