Question Test 1/1

Write answers on this side of the paper only.

Do not write on either margin

$\boxed{1}$ We use $[X, P] = i\hbar$ twice,

$$PX^3P - XPXPX = PX^3P - (PX + i\hbar)X(XP - i\hbar)$$

$$= PX^3P - (PX^3P - i\hbar PX^2 + i\hbar X^2P + \hbar^2 X)$$

$$= -i\hbar \underbrace{[X^2, P]}_{2i\hbar X} - \hbar^2 X = \underline{\underline{\hbar^2 X.}}$$

$\boxed{2}$ We have $f(X)^\dagger = \int dx \left(|x\rangle f(x) \langle x| \right)^\dagger = \int dx\, |x\rangle f(x)^* \langle x|$, so that

(a) $f(X) = f(X)^\dagger$ requires $f(x) = f(x)^*$, that is: $f(x)$ must be real for all x values;

(b) $f(X)^\dagger f(X) = 1$ requires $f(x)^* f(x) = |f(x)|^2 = 1$, that is: $f(x)$ must be a phase factor for all x values.

$\boxed{3}$ Since $e^{ipX/\hbar} e^{ixP/\hbar} = e^{-ixp/\hbar} e^{ixP/\hbar} e^{ipX/\hbar}$, we need $e^{-ixp/\hbar} = 1$, that is xp must be an integer multiple of $2\pi\hbar$: $xp = 0, \pm 2\pi\hbar, \pm 4\pi\hbar, \ldots$.

$\boxed{4}$ (a) To show that $UU^\dagger = 1$ we consider $\langle x | UU^\dagger | x' \rangle$,

$$\langle x | UU^\dagger | x' \rangle = \int dx'' \langle x | U | x'' \rangle \langle x'' | U^\dagger | x' \rangle$$

$$= \int dx'' \langle x | U | x'' \rangle \left(\langle x' | U | x'' \rangle \right)^*$$

$$= \int dx'' \frac{e^{ixx''/a^2}}{\sqrt{2\pi}\, a} \frac{e^{-ix'x''/a^2}}{\sqrt{2\pi}\, a}$$

$$= \frac{1}{2\pi a^2} \int dx''\, e^{i(x-x')x''/a^2}$$

$$= \frac{1}{2\pi a^2}\, 2\pi\, \delta\!\left(\frac{x-x'}{a^2}\right) = \delta(x-x'),$$

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so that $\langle x|UU^\dagger|x'\rangle = \langle x|\,1\,|x'\rangle$ for all $\langle x|$ and all $|x'\rangle$, implying $UU^\dagger = 1$ indeed.

$\boxed{4}$ (b) We have

$$\langle x|U|p'\rangle = \int dx' \,\langle x|U|x'\rangle\langle x'|p'\rangle$$

$$= \int dx' \,\frac{e^{ixx'/a^2}}{\sqrt{2\pi}\,a}\,\frac{e^{ix'p'/\hbar}}{\sqrt{2\pi\hbar}}$$

$$= \frac{1}{2\pi a\sqrt{\hbar}}\,2\pi\,\delta\left(\frac{x}{a^2}+\frac{p'}{\hbar}\right)$$

$$= \frac{1}{\sqrt{\hbar}}\,\delta(x/a + ap'/\hbar).$$

$\boxed{4}$ (c) Now we get

$$\langle p|U|p'\rangle = \int dx\,\langle p|x\rangle\langle x|U|p'\rangle$$

$$= \int dx\,\frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\frac{1}{\sqrt{\hbar}}\,\delta\left(\frac{x}{a}+\frac{ap'}{\hbar}\right)$$

$$= \frac{a}{\sqrt{2\pi}\,\hbar}\,e^{-ip\left(-\frac{a^2p'}{\hbar}\right)/\hbar}$$

$$= \frac{a/\hbar}{\sqrt{2\pi}}\,e^{i a^2 pp'/\hbar^2}.$$

$\boxed{4}$ (d) We have

$$\langle x|U^2|x'\rangle = \int dx''\,\langle x|U|x''\rangle\langle x''|U|x'\rangle$$

$$= \int dx''\,\frac{1}{2\pi a^2}\,e^{i(xx''+x''x')/a^2}$$

$$= \frac{1}{2\pi a^2}\,2\pi\,\delta\left(\frac{x+x'}{a^2}\right) = \delta(x+x')$$

$$= \langle -x|x'\rangle \quad \text{for all } |x'\rangle,$$

so that

$$\langle x|U^2 = \langle -x| \quad \text{for all } \langle x|.$$

Question Test 1/3

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$\boxed{5}$(a) $\langle e^{ikx} \rangle = \int dx \langle \,|x\rangle e^{ikx} \langle x|\, \rangle$

$$= \int dx\, e^{ikx} |\psi(x)|^2$$

$$= \int dx\, \underbrace{e^{ikx}}_{= \underbrace{\cos(kx)}_{\text{even}} + i \underbrace{\sin(kx)}_{\text{odd, does not contribute}}} \kappa\, e^{-2\kappa|x|}$$

$$= 2\kappa \int_0^\infty dx\, e^{-2\kappa|x|} \underbrace{\cos(kx)}_{= \mathrm{Re}\, e^{ikx}}$$

$$= 2\kappa\, \mathrm{Re} \frac{1}{2\kappa - ik} = \frac{(2\kappa)^2}{(2\kappa)^2 + k^2}.$$

$\boxed{5}$(b) It is $\langle x| e^{\frac{1}{2} iaP/\hbar} |\, \rangle = \langle x + \frac{a}{2} |\, \rangle = \psi(x + \frac{a}{2})$

and $\langle \,| e^{\frac{1}{2} iaP/\hbar} |x\rangle = \langle \,| x - \frac{a}{2} \rangle = \psi(x - \frac{a}{2})^*$,

so that

$$\langle e^{iaP/\hbar} \rangle = \int dx \langle \,| e^{\frac{1}{2} iaP/\hbar} |x\rangle \langle x| e^{\frac{1}{2} iaP/\hbar} |\, \rangle$$

$$= \int dx\, \psi(x - \tfrac{a}{2})^* \psi(x + \tfrac{a}{2})$$

$$= \kappa \int dx\, e^{-\kappa(|x - \frac{a}{2}| + |x + \frac{a}{2}|)}.$$

Since $|x - \frac{a}{2}| + |x + \frac{a}{2}|$ is even in both x and a, we can take twice the integral $\int_0^\infty dx$ and replace a by $|a|$. Then

$$\langle e^{iaP/\hbar} \rangle = 2\kappa \int_0^\infty dx\, e^{-\kappa(|x - \frac{|a|}{2}| + |x + \frac{|a|}{2}|)}$$

Question Test 1/4

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$$= 2\kappa \int_0^{|a|/2} dx\, e^{-\kappa|a|} + 2\kappa \int_{|a|/2}^{\infty} dx\, e^{-2\kappa x}$$

$$= \kappa|a|\, e^{-\kappa|a|} + e^{-\kappa|a|} = \underline{\underline{(1+\kappa|a|)\, e^{-\kappa|a|}}}.$$

$\boxed{5}$ (c) Upon expanding $\langle e^{ikX}\rangle = \frac{(2\kappa)^2}{(2\kappa)^2+k^2} = 1 - \frac{k^2}{(2\kappa)^2+k^2}$ in powers of k up to k^2,

$$\left\langle \left(1 + ikX - \tfrac{1}{2}k^2X^2\right)\right\rangle = 1 - \frac{k^2}{(2\kappa)^2},$$

we read off that $\langle X\rangle = 0$, $\langle X^2\rangle = \frac{1}{2\kappa^2}$.

Likewise

$$\langle e^{iaP/\hbar}\rangle = (1+\kappa|a|)\left(1 - \kappa|a| + \tfrac{1}{2}(\kappa a)^2 + \cdots\right)$$

gives $\left\langle\left(1 + i\frac{a}{\hbar}P - \frac{1}{2}\left(\frac{a}{\hbar}P\right)^2\right)\right\rangle = 1 - \frac{1}{2}(\kappa a)^2,$

which yields

$$\langle P\rangle = 0, \quad \langle P^2\rangle = (\hbar\kappa)^2.$$

The spreads $\delta X = \frac{1}{\sqrt{2}\,\kappa}$, $\delta P = \hbar\kappa$ follow immediately, and their product obeys the uncertainty relation

$$\delta X\, \delta P = \frac{\hbar}{\sqrt{2}} = \sqrt{2}\,\frac{\hbar}{2} > \frac{\hbar}{2},$$

as it should.