Question. Test 2/1

Write answers on this side of the paper only.

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$\boxed{1}$ We have

$$\frac{d}{dt} P(t) = 0, \quad P(t) = P(t_0)$$

and $$\frac{d}{dt} X(t) = \frac{1}{M(t)} P(t) = \frac{1}{M(t)} P(t_0),$$

so that $$X(t) = X(t_0) + P(t_0) \int_{t_0}^{t} \frac{dt'}{M(t')}.$$

It follows that

$$i\hbar \frac{\partial}{\partial t} \langle x, t | p, t_0 \rangle = \langle x, t | \, H(t) \, | p, t_0 \rangle$$

$$= \langle x, t | \, \frac{1}{2M(t)} P(t_0)^2 \, | p, t_0 \rangle$$

$$= \frac{p^2}{2M(t)} \langle x, t | p, t_0 \rangle ;$$

$$-i\hbar \frac{\partial}{\partial t_0} \langle x, t | p, t_0 \rangle = \langle x, t | \, H(t_0) \, | p, t_0 \rangle$$

$$= \langle x, t | \, \frac{1}{2M(t_0)} P(t_0)^2 \, | p, t_0 \rangle$$

$$= \frac{p^2}{2M(t_0)} \langle x, t | p, t_0 \rangle ;$$

$$\frac{\hbar}{i} \frac{\partial}{\partial x} \langle x, t | p, t_0 \rangle = \langle x, t | \, P(t) \, | p, t_0 \rangle$$

$$= \langle x, t | \, P(t_0) \, | p, t_0 \rangle$$

$$= p \, \langle x, t | p, t_0 \rangle ;$$

$$\frac{\hbar}{i} \frac{\partial}{\partial p} \langle x, t | p, t_0 \rangle = \langle x, t | \, X(t_0) \, | p, t_0 \rangle$$

$$= \langle x, t | \left(X(t) - P(t) \int_{t_0}^{t} \frac{dt'}{M(t')} \right) | p, t_0 \rangle$$

$$= \left(x - p \int_{t_0}^{t} \frac{dt'}{M(t')} \right) \langle x, t | p, t_0 \rangle .$$

Question. Test 2/2

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Taken together these differential equations imply

$$\langle x, t | p, t_0 \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ixp/\hbar} e^{-\frac{i}{\hbar}\frac{p^2}{2}\int_{t_0}^{t}\frac{dt'}{M(t')}},$$

with the prefactor determined by the $t \to t_0$ limit, or the familiar $M(t) = \text{const}$ result.

$\boxed{2}$ We have $V^\dagger = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$, so that

$$V^\dagger V = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |m\rangle \underbrace{\langle m+1|n+1\rangle}_{\delta_{n,m}} \langle n| = \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$= 1$, indeed.

But

$$VV^\dagger = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |n+1\rangle \underbrace{\langle n|m\rangle}_{\delta_{nm}} \langle m+1|$$

$$= \sum_{n=0}^{\infty} |n+1\rangle\langle n+1| = \sum_{n=1}^{\infty} |n\rangle\langle n|$$

$$= 1 - |0\rangle\langle 0| \neq 1.$$

In particular, $\langle n=0| V = 0$, which is impossible if V were unitary.

Further,

$$V = A^\dagger \sum_{n=0}^{\infty} |n\rangle \frac{1}{\sqrt{n+1}} \langle n|$$

$$= A^\dagger \frac{1}{\sqrt{A^\dagger A + 1}},$$

so that $W(A^\dagger A) = (A^\dagger A + 1)^{-1/2}$.

Question. Test 2/3

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[3] We know that $|n\rangle\langle n| = \frac{A^{+n}}{\sqrt{n!}} |0\rangle\langle 0| \frac{A^n}{\sqrt{n!}}$

$$= \frac{1}{n!} A^{+n} e^{-A^+;A} A^n$$

(see the Exercise on page 112), so that

$$F(A^+A) = \sum_{n=0}^{\infty} |n\rangle F(n) \langle n|$$

$$= \sum_{n=0}^{\infty} \frac{F(n)}{n!} A^{+n} e^{-A^+;A} A^n,$$

indeed.

Now, for $F(A^+A) = z^{A^+A}$,

$$z^{A^+A} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n A^{+n} e^{-A^+;A} A^n$$

$$= \left(e^{zA^+A} e^{-A^+A}\right)_{\text{normally ordered}}$$

$$= e^{(z-1)A^+;A}.$$

[4] Writing

$$H = \left(\frac{1}{2M} P_1^2 + \frac{1}{2} M\omega^2 X_1^2\right) + \left(\frac{1}{2M} P_2^2 + \frac{1}{2} M\omega^2 X_2^2\right)$$
$$+ \left(\frac{1}{2M} P_3^2 + \frac{1}{2} M\omega^2 X_3^2\right)$$

we see that the joint eigenstates of the three individual one-dimensional harmonic oscillators are eigenstates of H, so that

$$H|n_1, n_2, n_3\rangle = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right),$$

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or $\quad H|n_1, n_2, n_3\rangle = |n_1, n_2, n_3\rangle \hbar\omega\left(N + \frac{3}{2}\right)$

with $N = n_1 + n_2 + n_3$. To given N value we have

$$n_1 = 0, 1, 2, \ldots, N;$$
$$n_2 = 0, 1, \ldots, N - n_1;$$
$$n_3 = N - n_1 - n_2 .$$

So for each choice of both n_1 and n_2, there is just one value of n_3, one possible ket $|n_1, n_2, n_3\rangle$. And for a given value of n_1, we have $N - n_1 + 1$ possible values of n_2. The total count of kets $|n_1, n_2, n_3\rangle$ with given $N = n_1 + n_2 + n_3$ is therefore

$$\sum_{n_1=0}^{N} (N - n_1 + 1) = \sum_{m=1}^{N+1} m = \underline{\underline{\tfrac{1}{2}(N+1)(N+2)}}.$$

$\boxed{5}$ Using $\quad L_1 = \frac{1}{2}(L_+ + L_-)$, $\quad L_2 = \frac{1}{2i}(L_+ - L_-)$, we have

$$L_1|\rangle = |l=1, m=1\rangle \frac{\hbar}{3\sqrt{2}} + |l=1, m=0\rangle \frac{\sqrt{8}\,\hbar}{3}$$
$$+ |l=1, m=-1\rangle \frac{\hbar}{3\sqrt{2}} ,$$

$$L_2|\rangle = |l=1, m=1\rangle \frac{\hbar}{i3\sqrt{2}} + |l=1, m=-1\rangle \frac{i\hbar}{3\sqrt{2}} ,$$

so that

$$\langle L_1 \rangle = \frac{2}{3}\frac{\hbar}{3\sqrt{2}} + \frac{1}{3}\frac{\sqrt{8}\,\hbar}{3} + \frac{2}{3}\frac{\hbar}{3\sqrt{2}} = \frac{4\sqrt{2}}{9}\hbar$$

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and $\langle L_2 \rangle = \frac{2}{3}\frac{\hbar}{i3\sqrt{2}} + \frac{2}{3}\frac{i\hbar}{3\sqrt{2}} = 0$

as well as

$$\langle L_1^2 \rangle = \left(\frac{\hbar}{3\sqrt{2}}\right)^2 + \left(\frac{\sqrt{8}\hbar}{3}\right)^2 + \left(\frac{\hbar}{3\sqrt{2}}\right)^2 = \hbar^2$$

and $\langle L_2^2 \rangle = \left(\frac{\hbar}{3\sqrt{2}}\right)^2 + \left(\frac{\hbar}{3\sqrt{2}}\right)^2 = \frac{1}{9}\hbar^2$,

giving $\delta L_1 = \sqrt{\hbar^2 - \frac{32}{81}\hbar^2} = \frac{7}{9}\hbar$

and $\delta L_2 = \sqrt{\frac{1}{9}\hbar^2 - 0} = \frac{1}{3}\hbar$.

For L_3 we find immediately that

$$\langle L_3 \rangle = \left(\frac{2}{3}\right)^2 \hbar + \left(\frac{1}{3}\right)^2 0\hbar + \left(\frac{2}{3}\right)^2(-\hbar) = 0,$$

$$\langle L_3^2 \rangle = \left(\frac{2}{3}\right)^2 \hbar^2 + \left(\frac{1}{3}\right)^2 (0\hbar)^2 + \left(\frac{2}{3}\right)^2(-\hbar)^2 = \frac{8}{9}\hbar^2,$$

giving $\delta L_3 = \frac{2\sqrt{2}}{3}\hbar$.

The uncertainty relation for L_1 and L_2 is

$$\delta L_1 \, \delta L_2 \geq \frac{1}{2}\left| i \langle \underbrace{[L_1, L_2]}_{i\hbar L_3} \rangle \right| = \frac{\hbar}{2}\left|\langle L_3 \rangle\right|,$$

here:

$$\frac{7}{9}\hbar \, \frac{1}{3}\hbar \geq \frac{\hbar}{2} 0, \quad \text{OK.}$$

Likewise $\delta L_2 \delta L_3 \geq \frac{\hbar}{2} |\langle L_1 \rangle|$,

here:

$$\frac{1}{3}\hbar \, \frac{2\sqrt{2}}{3}\hbar \geq \frac{\hbar}{2} \frac{4\sqrt{2}}{9}\hbar, \quad \text{OK with = sign.}$$

Finally $\delta L_3 \delta L_1 \geq \frac{\hbar}{2} |\langle L_2 \rangle|$,

here: $\frac{2\sqrt{2}}{3}\hbar \, \frac{7}{9}\hbar \geq \frac{\hbar}{2} \, 0$, OK.