(1) $H = \hbar w(t) A^{+}(t) A(t)$

$$\frac{d}{dt} A(t) = -iw(t) A(t)$$

$$A(t) = A(t_0) e^{-i\int_{t_0}^{t} dt' w(t')}$$

$$\frac{d}{dt} A^{+}(t) = iw(t) A^{+}(t)$$

$$A^{+}(t) = A^{+}(t_0) e^{i\int_{t_0}^{t} dt' w(t')}$$

$$i\hbar \frac{\partial}{\partial t} \langle a^*, t | a', t_0 \rangle = \langle a^*, t | H(t) | a', t_0 \rangle$$

$$= \langle a^*, t | A^{+}(t) A(t_0) e^{-i\int_{t_0}^{t} dt' w(t')} | a', t_0 \rangle \hbar w(t)$$

$$= a^* a' e^{-i\int_{t_0}^{t} dt' w(t')} \hbar w(t) \ \langle a^*, t | a', t_0 \rangle$$

$$i\hbar \frac{\partial}{\partial t} \log \langle a^*, t | a', t_0 \rangle = \hbar w(t) a^* a' e^{-i\int_{t_0}^{t} dt' w(t')}$$

$$= \frac{\partial}{\partial t} (i\hbar) a^* a' e^{-i\int_{t_0}^{t} dt' w(t')} \qquad \text{using the chain rule}$$

$$\log \langle a^*, t | a', t_0 \rangle = a^* a' e^{-i\int_{t_0}^{t} dt' w(t')} + c$$

$$\langle a^*, t | a', t_0 \rangle = C e^{a^* a' e^{-i\int_{t_0}^{t} dt' w(t')}}$$

The prefactor can be found by taking the limit $t \to t_0$,

$\therefore C = 1$

$$\therefore \langle a^*, t | a', t_0 \rangle = e^{a^* a' e^{-i\int_{t_0}^{t} dt' w(t')}} \quad //$$

These sample solutions were prepared by Hong Zhenxi.

Q2) To ensure that the transformation is unitary,

$(\lambda_1 X + \mu_1 P)$ and $(\lambda_2 P + \mu_2 X)$ must continue to obey the commutation relations.

$\Rightarrow [X, P] = i\hbar \longrightarrow [(\lambda_1 X + \mu_1 P), (\lambda_2 P + \mu_2 X)] = [\lambda_1 X, (\lambda_2 P + \mu_2 X)] + [\mu_1 P, (\lambda_2 P + \mu_2 X)]$

$= [\lambda_1 X, \lambda_2 P] + [\lambda_1 X, \mu_2 X] + [\mu_1 P, \lambda_2 P] + [\mu_1 P, \mu_2 X]$

$= \lambda_1\lambda_2 i\hbar - \mu_1\mu_2 i\hbar$

$= i\hbar(\lambda_1\lambda_2 - \mu_1\mu_2)$

In order for the transformed operators to obey the commutation relation $[X, P] = i\hbar$,

$\lambda_1\lambda_2 - \mu_1\mu_2 = 1$

$\therefore \lambda_1\lambda_2 = \mu_1\mu_2 + 1$ // — (1)

Also, after transformation, operators must remain hermitian.

$\therefore \lambda_1 X + \mu_1 P = (\lambda_1 X + \mu_1 P)^\dagger$

$= \lambda_1^* X + \mu_1^* P$

$\Rightarrow \lambda_1 = \lambda_1^*$; $\mu_1 = \mu_1^*$

$\therefore \lambda_1$ and μ_1 must be real.

Similarly, $(\lambda_2 P + \mu_2 X) = (\lambda_2 P + \mu_2 X)^\dagger$

$= \lambda_2^* P + \mu_2^* X$

$\therefore \lambda_2$ and μ_2 must also be real

$$\begin{pmatrix} X \\ P \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu_2 & \lambda_2 \end{pmatrix}\begin{pmatrix} X \\ P \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1' & 0 \\ \mu_2' & \lambda_2' \end{pmatrix}\begin{pmatrix} \lambda_1'' & \mu_1'' \\ 0 & \lambda_2'' \end{pmatrix} = \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu_2 & \lambda_2 \end{pmatrix}$$

Looking at matrix $\begin{pmatrix} \lambda_1' & 0 \\ \mu_2' & \lambda_2' \end{pmatrix}$,

$\begin{pmatrix} \lambda_1' & 0 \\ \mu_2' & \lambda_2' \end{pmatrix} = \lambda_1'\lambda_2' = 1$ since $\mu_1'\mu_2' = 0$ in this case (using property (1))

$\therefore \lambda_2' = \frac{1}{\lambda_1'}$

Looking at matrix $\begin{pmatrix} \lambda_1'' & \mu_1'' \\ 0 & \lambda_2'' \end{pmatrix}$,

$\begin{pmatrix} \lambda_1'' & \mu_1'' \\ 0 & \lambda_2'' \end{pmatrix} = \lambda_1''\lambda_2'' = 1$ since $\mu_1''\mu_2'' = 0$ in this case (using property (2))

$\therefore \lambda_2'' = \frac{1}{\lambda_1''}$

Q3)

$$F = \sum_{k=0}^{\infty} \frac{f_k}{k!} A^{+k} A^k$$

$$= \sum_{k=0}^{\infty} \frac{f_k}{k!} \frac{(A^+A)!}{(A^+A - k)!}$$ from lecture notes Pg 126

$$= \sum_{k=0}^{\infty} \binom{A^+A}{k} f_k$$ // [shown]

For $f_k = y^k$,

$$F = \sum_{k=0}^{\infty} \binom{A^+A}{k} y^k$$

$$= (1+y)^{A^+A}$$ // using $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

For example

$$\begin{pmatrix} \lambda_1 & 0 \\ \mu_2 & 1/\lambda_1 \end{pmatrix} \begin{pmatrix} 1 & \mu_1/\lambda_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu_2 & (\mu_1\mu_2+1)/\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu_2 & \lambda_2 \end{pmatrix}$$

works whenever $\lambda_1 \neq 0$, and

$$\begin{pmatrix} 1/\lambda_2 & \mu_1 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mu_2/\lambda_2 & 1 \end{pmatrix} = \begin{pmatrix} (\mu_1\mu_2+1)/\lambda_2 & \mu_1 \\ \mu_2 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \mu_1 \\ \mu_2 & \lambda_2 \end{pmatrix}$$

works whenever $\lambda_2 \neq 0$. If both are $\neq 0$, either order of multiplication is applicable.

(4) $H = \frac{1}{2M}(P_1^2 + P_2^2) + \frac{1}{2}Mw^2(x_1^2 + x_2^2) + \frac{1}{2}w(x_1P_2 - x_2P_1)$

$= \hbar w(A_1^+A_1 + A_2^+A_2) + \hbar w + \frac{1}{2}wL_3$

$= \hbar w(A_+^+A_+ + A_-^+A_-) + \hbar w + \frac{1}{2}\hbar w(A_+^+A_+ - A_-^+A_-)$

$H|n_+, n_-\rangle = [\hbar w(A_+^+A_+ + A_-^+A_-) + \hbar w + \frac{1}{2}\hbar w(A_+^+A_+ - A_-^+A_-)]|n_+, n_-\rangle$

$= |n_+, n_-\rangle[\hbar w(n_+ + n_-) + \hbar w + \frac{1}{2}\hbar w(n_+ - n_-)]$

$= |n_+, n_-\rangle \hbar w(n_+ + n_- + 1 + \frac{1}{2}n_+ - \frac{1}{2}n_-)$

$= |n_+, n_-\rangle \hbar w(\frac{3}{2}n_+ + \frac{1}{2}n_- + 1) = |n_+, n_-\rangle \frac{\hbar w}{2}(3n_+ + n_- + 2) = |n_+, n_-\rangle \frac{\hbar w}{2}N$

Possible values of $3n_+$: $0, 1, 2 \ldots N-2$

Possible values of n_-: $N-2-3n_+$

Given N	Multiplicity
2	1
3	1
4	1
5	2
6	2
7	2
8	3

$\therefore$ multiplicities $= \lfloor \frac{N+1}{3} \rfloor$ where $\lfloor \ \rfloor \equiv$ the largest integer not exceeding number in bracket

(5) $L_+ f(L_3) = (L_1 + iL_2) f(L_3)$

$= f(L_3 - \hbar)(L_1 + iL_2)$

$L_- f(L_3) = (L_1 - iL_2) f(L_3) = f(L_3 + \hbar)(L_1 - iL_2)$

$(L_1 + iL_3) f(L_2) = f(L_2 + \hbar)(L_1 + iL_3)$ since $[L_1, L_3] = -i\hbar L_2$

$(L_1 - iL_3) f(L_2) = f(L_2 - \hbar)(L_1 - iL_3)$ since $[L_1, L_3] = -i\hbar L_2$

$\therefore (L_1 \pm iL_3) f(L_2) = f(L_2 \pm \hbar)(L_1 \pm iL_3)$ //

Let $L_+ = L_1 + iL_3$, $L_- = L_1 - iL_3$ in this case.

$L_1 = \frac{1}{2}(L_+ + L_-)$

$$e^{\frac{1}{2}i\pi L_2/\hbar} L_1 e^{-\frac{1}{2}i\pi L_2/\hbar} = e^{\frac{1}{2}i\pi L_2/\hbar} \frac{1}{2}L_+ e^{-\frac{1}{2}i\pi L_2/\hbar} + e^{\frac{1}{2}i\pi L_2/\hbar} \frac{1}{2}L_- e^{-\frac{1}{2}i\pi L_2/\hbar}$$

$$= \frac{1}{2}e^{\frac{1}{2}i\pi L_2/\hbar} e^{-\frac{1}{2}i\pi(L_2+\hbar)/\hbar} L_+ + \frac{1}{2}e^{\frac{1}{2}i\pi L_2/\hbar} e^{-\frac{1}{2}i\pi(L_2-\hbar)/\hbar} L_-$$

$$= \frac{1}{2}e^{-\frac{1}{2}i\pi} L_+ + \frac{1}{2}e^{\frac{1}{2}i\pi} L_-$$

$$= \frac{1}{2}(\cos\tfrac{\pi}{2} - i\sin\tfrac{\pi}{2}) L_+ + \frac{1}{2}(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}) L_-$$

$$= -\frac{1}{2}iL_+ + \frac{1}{2}iL_-$$

$$= -\frac{1}{2}i(L_+ - L_-)$$

$$= -\frac{1}{2}i(2iL_3)$$

$$= L_3$$

$$e^{\frac{1}{2}i\pi L_2/\hbar} L_1 = L_3 e^{\frac{1}{2}i\pi L_2/\hbar}$$

$$\langle l,m| e^{\frac{1}{2}i\pi L_2/\hbar} L_1 = \langle l,m| L_3 e^{\frac{1}{2}i\pi L_2/\hbar}$$

$$= m\hbar \langle l,m| e^{\frac{1}{2}i\pi L_2/\hbar}$$ // [shown]