Problem 1 (20 marks)
Harmonic oscillator with ladder operators A, $A^\dagger$: Determine the time transformation function $\langle a^*, t|a', t_0\rangle$ for the Hamilton operator

$$H = \hbar\omega(t)A^\dagger A\,,$$

which has a time-dependent frequency $\omega(t)$.

Problem 2 (20 marks)
Motion along the x axis: A linear transformation of position operator X and momentum operator P is given by

$$X \to \lambda_1 X + \mu_1 P\,, \qquad P \to \lambda_2 P + \mu_2 X\,.$$

State all properties of the numerical coefficients λ_1, λ_2, μ_1, and μ_2 that are necessary to ensure that the transformation is unitary. Then show that all such unitary transformations can be realized as two successive transformations of this kind with $\mu_1 = 0$ for one of the two and $\mu_2 = 0$ for the other.

Problem 3 (20 marks)
Harmonic oscillator with ladder operators A, $A^\dagger$: Show that

$$F = \sum_{k=0}^{\infty} \binom{A^\dagger A}{k} f_k\,,$$

if the normally-ordered form of an operator F has the form

$$F = \sum_{k=0}^{\infty} \frac{f_k}{k!} A^{\dagger k} A^k$$

with complex coefficients f_k. Use this for $f_k = y^k$ to express the corresponding F compactly as a function of $A^\dagger A$.

Problem 4 (20 marks)
Two-dimensional motion: What are the eigenvalues and their multiplicities of the Hamilton operator

$$H = \frac{1}{2M}(P_1^2 + P_2^2) + \frac{1}{2}M\omega^2(X_1^2 + X_2^2) + \frac{1}{2}\omega(X_1P_2 - X_2P_1)\,,$$

where the mass M and the frequency ω are positive constants?

Problem 5 (20 marks)
Orbital angular momentum with $\langle l, m|$ denoting the usual common eigenbras of $\vec{L}^2$ and L_3: Show first that

$$(L_1 \pm \mathrm{i}L_3)f(L_2) = f(L_2 \pm \hbar)(L_1 \pm \mathrm{i}L_3)$$

for any function $f(L_2)$ of L_2, and then use this to demonstrate that

$$\langle l, m|\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi L_2/\hbar}$$

is an eigenbra of L_1 with eigenvalue $m\hbar$.