AY 2005/06, Sem 2, PC3130, Sample solutions to the Exam on 25 April 2006

Question. . . . .

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$\boxed{\text{I}}$ (a) We need to verify that $U^\dagger U = 1$ and $UU^\dagger = 1$ with U as given and

$$U^\dagger = \int_{-\infty}^{\infty} d\lambda' \, |x=\lambda' x_0\rangle \sqrt{x_0 p_0} \langle p = \lambda' p_0| .$$

$$U^\dagger U = \int d\lambda' \, d\lambda \; |x=\lambda' x_0\rangle \sqrt{x_0 p_0}^{\,2} \underbrace{\delta(\lambda' p_0 - \lambda p_0)}_{=\frac{1}{p_0}\delta(\lambda'-\lambda)} \langle x = \lambda x_0|$$

$$= \int d\lambda' \; |x=\lambda' x_0\rangle \, x_0 \, \langle x = \lambda' x_0|$$

$$= \int dx \; |x\rangle\langle x| = 1 .$$

$$UU^\dagger = \int d\lambda \, d\lambda' \, |p=\lambda p_0\rangle \sqrt{x_0 p_0}^{\,2} \underbrace{\delta(\lambda x_0 - \lambda' x_0)}_{=\frac{1}{x_0}\delta(\lambda-\lambda')} \langle p = \lambda' p_0|$$

$$= \int d\lambda \; |p=\lambda p_0\rangle \, p_0 \, \langle p = \lambda p_0|$$

$$= \int dp \; |p\rangle\langle p| = 1.$$

Indeed, U is unitary.

$\boxed{\text{I}}$ (b)

$$U|x\rangle = \int d\lambda \; |p=\lambda p_0\rangle \sqrt{p_0 x_0} \, \underbrace{\delta(\lambda x_0 - x)}_{=\frac{1}{x_0}\delta(\lambda - x/x_0)}$$

$$= |p = p_0 x/x_0\rangle \sqrt{p_0/x_0} .$$

$\boxed{\text{I}}$ (c)

$$\langle p|U = \int d\lambda \; \delta(p - \lambda p_0) \sqrt{p_0 x_0} \langle x = \lambda x_0|$$

$$= \sqrt{x_0/p_0} \; \langle x = x_0 p/p_0| ,$$ <u>yes</u>!

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1 (d) We need to show that $U^{+} X U = - x_0 P/p_0$ and $U^{+} P U = p_0 X/x_0$, or

$$p_0 X U = - U x_0 P \quad \text{and} \quad x_0 P U = U p_0 X.$$

$$x_0 P U = \int_{-\infty}^{\infty} d\lambda \; x_0 P |p=\lambda p_0\rangle \sqrt{p_0 x_0} \langle x=\lambda x_0|$$

$$= \int d\lambda \; |p=\lambda p_0\rangle \lambda x_0 p_0 \sqrt{p_0 x_0} \langle x=\lambda x_0|$$

$$= \int d\lambda \; |p=\lambda p_0\rangle \sqrt{p_0 x_0} \langle x=\lambda x_0| p_0 X$$

$$= U p_0 X, \text{ indeed.}$$

$$p_0 X U = \int d\lambda \; p_0 X |p=\lambda p_0\rangle \sqrt{p_0 x_0} \langle x=\lambda x_0|$$

$$= \int d\lambda \; \frac{\hbar}{i} \frac{\partial |p=\lambda p_0\rangle}{\partial\lambda} \sqrt{p_0 x_0} \langle x=\lambda x_0|$$

integrate by parts

$$\stackrel{\downarrow}{=} - \int d\lambda \; |p=\lambda p_0\rangle \sqrt{p_0 x_0} \frac{\hbar}{i} \frac{\partial \langle x=\lambda x_0|}{\partial\lambda}$$

$$= - \int d\lambda \; |p=\lambda p_0\rangle \sqrt{p_0 x_0} \langle x=\lambda x_0| x_0 P$$

$$= - U x_0 P, \text{ correct as well.}$$

The rest follows from

$$U^{+} f(X,P) U = f(U^{+}XU, U^{+}PU)$$

$$= f(-x_0 P/p_0, \; p_0 X/x_0).$$

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$\boxed{2}$(a) $\frac{d}{dt} X_1 = \frac{\partial H}{\partial P_1} = \frac{1}{M} P_1 - \omega X_2$,

$\frac{d}{dt} X_2 = \frac{\partial H}{\partial P_2} = \frac{1}{M} P_2 + \omega X_1$,

$\frac{d}{dt} P_1 = -\frac{\partial H}{\partial X_1} = -\omega P_2$,

$\frac{d}{dt} P_2 = -\frac{\partial H}{\partial X_2} = \omega P_1$.

$\boxed{2}$(b) $\frac{d}{dt}(P_1^2 + P_2^2) = \frac{dP_1}{dt} P_1 + P_1 \frac{dP_1}{dt}$

$+ \frac{dP_2}{dt} P_2 + P_2 \frac{dP_2}{dt}$

$= -\omega P_2 P_1 - \omega P_1 P_2$

$+ \omega P_1 P_2 + \omega P_2 P_1 = 0$,

$\frac{d}{dt}(X_1 P_2 - X_2 P_1) = \frac{dX_1}{dt} P_2 + X_1 \frac{dP_2}{dt}$

$- \frac{dX_2}{dt} P_1 - X_2 \frac{dP_1}{dt}$

$= \frac{1}{M}(P_1 - M\omega X_2) P_2 + \omega X_1 P_1$

$- \frac{1}{M}(P_2 + M\omega X_1) P_1 + \omega X_2 P_2$

$= \frac{1}{M}[P_1, P_2] = 0$.

$\boxed{2}$(c) $P_1(t) = P_1(t_0)\cos\phi - P_2(t_0)\sin\phi$,

$P_2(t) = P_2(t_0)\cos\phi + P_1(t_0)\sin\phi$

with $\phi = \omega(t - t_0)$.

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$\boxed{2}$ (d) In $i\hbar \frac{\partial}{\partial t}\langle x_1, x_2, t | p_1, p_2, t_0\rangle$

$$= \langle x_1, x_2, t | H(t) | p_1, p_2, t_0\rangle$$

we need to express

$$H(t) = \frac{1}{2M}\left[P_1(t)^2 + P_2(t)^2\right] + \omega\left[X_1(t)P_2(t) - X_2(t)P_1(t)\right]$$

in terms of $X_1(t)$, $X_2(t)$ and $P_1(t_0)$, $P_2(t_0)$:

$$H(t) = \frac{1}{2M}\left[P_1(t_0)^2 + P_2(t_0)^2\right]$$
$$+ \omega\, X_1(t)\left(P_2(t_0)\cos\phi + P_1(t_0)\sin\phi\right)$$
$$- \omega\, X_2(t)\left(P_1(t_0)\cos\phi - P_2(t_0)\sin\phi\right),$$

giving

$$i\hbar\frac{\partial}{\partial t}\langle x_1, x_2, t | p_1, p_2, t_0\rangle$$
$$= \left[\frac{p_1^2 + p_2^2}{2M} + \omega(x_1p_2 - x_2p_1)\cos\phi\right.$$
$$\left. + \omega(x_1p_1 + x_2p_2)\sin\phi\right]\langle x_1, x_2, t | p_1, p_2, t_0\rangle$$

where $[\cdots] = \frac{\partial}{\partial t}\left(\frac{t-t_0}{2M}(p_1^2+p_2^2) + (x_1p_2 - x_2p_1)\sin\phi\right.$
$\left. - (x_1p_1 + x_2p_2)\cos\phi\right)$,

so that

$$\langle x_1, x_2, t | p_1, p_2, t_0\rangle = \frac{e^{\frac{i}{\hbar}(x_1p_1 + x_2p_2)\cos\phi}}{2\pi\hbar}$$
$$\cdot e^{-\frac{i}{\hbar}\frac{t-t_0}{2M}(p_1^2+p_2^2) - \frac{i}{\hbar}(x_1p_2 - x_2p_1)\sin\phi}$$

with the factor $\frac{1}{2\pi\hbar}$ ensuring the correct limit of $t \to t_0$ or the correct limit of $\omega \to 0$.

3(a) Since $L_+|\rangle = |l=1, m=0\rangle \hbar\sqrt{2}\,\beta$,

$L_-|\rangle = |l=1, m=0\rangle \hbar\sqrt{2}\,\alpha$

we have

$$L_1|\rangle = |l=1, m=0\rangle \frac{\alpha+\beta}{\sqrt{2}}\hbar,$$

$$L_2|\rangle = |l=1, m=0\rangle \frac{i(\alpha-\beta)}{\sqrt{2}}\hbar,$$

and $L_3|\rangle = |l=1, m=1\rangle \alpha\hbar - |l=1, m=-1\rangle \beta\hbar$,

so that

$$\langle L_1\rangle = 0, \quad \langle L_2\rangle = 0, \quad \langle L_3\rangle = (|\alpha|^2 - |\beta|^2)\hbar,$$

and

$$\langle L_1^2\rangle = \tfrac{1}{2}|\alpha+\beta|^2\hbar^2, \quad \langle L_2^2\rangle = \tfrac{1}{2}|\alpha-\beta|^2\hbar^2,$$

$$\langle L_3^2\rangle = (|\alpha|^2+|\beta|^2)\hbar^2 = \hbar^2.$$

It follows that

$$\delta L_1 = \frac{1}{\sqrt{2}}|\alpha+\beta|\hbar, \quad \delta L_2 = \frac{1}{\sqrt{2}}|\alpha-\beta|\hbar,$$

$$\delta L_3 = \sqrt{(|\alpha|^2+|\beta|^2)^2 - (|\alpha|^2-|\beta|^2)^2}\;\hbar = 2|\alpha\beta|\hbar.$$

3(b)

$$\delta L_1\,\delta L_2 \geq \frac{\hbar}{2}|\langle L_3\rangle|,$$

$$\delta L_2\,\delta L_3 \geq \frac{\hbar}{2}|\langle L_1\rangle|,$$

$$\delta L_3\,\delta L_1 \geq \frac{\hbar}{2}|\langle L_2\rangle|,$$

of which the second and third are surely obeyed here since both right-hand sides vanish.

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$\boxed{3}$(c) For $\alpha = \frac{3}{5}$, $\beta = \frac{4}{5}$ we have

$$\delta L_1 = \frac{7\hbar}{5\sqrt{2}}, \quad \delta L_2 = \frac{\hbar}{5\sqrt{2}}, \quad \langle L_3 \rangle = -\frac{7\hbar}{25},$$

so that

$$\delta L_1 \delta L_2 = \frac{7\hbar^2}{50} = \frac{\hbar}{2} |\langle L_3 \rangle|, \text{ indeed.}$$

$\boxed{3}$(d) We need $\frac{\hbar}{\sqrt{2}} |\alpha+\beta| \frac{\hbar}{\sqrt{2}} |\alpha-\beta| = \frac{\hbar}{2} \left| |\alpha|^2 - |\beta|^2 \right| \hbar$

or

$$|\alpha+\beta| \, |\alpha-\beta| = \left| |\alpha^2| - |\beta|^2 \right|.$$

Now, since

$$\begin{aligned} |\alpha+\beta| \, |\alpha-\beta| &= \left| (\alpha+\beta)^* (\alpha-\beta) \right| \\ &= \Big| \underbrace{(\alpha^*\alpha - \beta^*\beta)}_{\text{real part}} - \underbrace{(\alpha^*\beta - \beta^*\alpha)}_{\text{imaginary part}} \Big| \\ &\geq \left| (\text{real part}) \right| = \left| |\alpha|^2 - |\beta|^2 \right| \end{aligned}$$

where the equal sign only applies if the imaginary part vanishes, we need that this is the case: $\underline{\alpha^*\beta = \beta^*\alpha}$.

So the coefficients α, β must be such that $\underline{\alpha^*\beta \text{ is real}}$.

For the example in $\boxed{3}$(c) this is obviously the case.

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$\boxed{4}$(a) The unperturbed states are the Fock states to the ladder operator A, A^+ and therefore

$$\langle n^{(0)} | H_1 | n^{(0)} \rangle$$
$$= \hbar\Omega \langle n^{(0)} | (A^+AA^+ + AA^+A) | n^{(0)} \rangle$$
$$= 0 \text{ for all } n$$

because both summands have an odd number of ladder operators. That is

$$E_n^{(1)} = 0 \text{ for all } n.$$

$\boxed{4}$(b) In $E_n^{(2)} = -\sum_{m(\neq n)} \frac{|\langle m^{(0)} | H_1 | n^{(0)} \rangle|^2}{E_m^{(0)} - E_n^{(0)}}$

we need $E_m^{(0)} - E_n^{(0)} = (m-n)\hbar\omega$ and

$$\langle m^{(0)} | H_1 | n^{(0)} \rangle = \hbar\Omega \; m \langle m^{(0)} | A^+ | n^{(0)} \rangle + \hbar\Omega \langle m^{(0)} | A | n^{(0)} \rangle n$$
$$= \hbar\Omega \left(m\sqrt{n+1}\, \delta_{m,n+1} + n\sqrt{m+1}\, \delta_{m+1,n} \right),$$

giving

$$E_n^{(2)} = -\hbar\omega \left(\frac{\Omega}{\omega}\right)^2 \sum_{m(\neq n)} \frac{m^2(n+1)\delta_{m,n+1} + n^2(m+1)\delta_{m+1,n}}{m-n}$$
$$= -\hbar\omega \left(\frac{\Omega}{\omega}\right)^2 \left[(n+1)^3 - n^3\right]$$

or $E_n^{(2)} = -\hbar\omega \left(\frac{\Omega}{\omega}\right)^2 \left(3n^2 + 3n + 1\right).$

$\boxed{4}$(c) Since $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)}$
$$= \hbar\omega \left(n - (3n^2+3n+1)\left(\frac{\Omega}{\omega}\right)^2 \right)$$

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up to 2nd order, we have

$$\Delta E_n = E_n - E_{n-1}$$
$$= \hbar\omega \left(1 - \left[(n+1)^3 - 2n^3 + (n-1)^3\right]\left(\frac{\Omega}{\omega}\right)^2\right)$$
$$= \hbar\omega \left(1 - 6n\left(\frac{\Omega}{\omega}\right)^2\right).$$

$\boxed{4}$ (d) Clearly if $\Delta E_n < 0$, the 2nd-order approximation is not trustworthy. Accordingly, higher-order terms must surely be included, when

$$n > \frac{1}{6}\left(\frac{\omega}{\Omega}\right)^2$$

and the approximation can be unreliable unless

$$n \ll \frac{1}{6}\left(\frac{\omega}{\Omega}\right)^2.$$