

PC 3274, AY 2006/07-I, Test 1

Question .....1/5.....

Write answers on this side of the paper only.

Do not write on either margin

□ By the usual procedure we get

$$xe^z dx = yz dy,$$

$$2xy dy = xe^z dz,$$

$$yz dz = 2xy dx,$$

$$\text{or: } d(y^2 - e^z) = 0,$$

$$\text{or: } d(z^2 - 2x^2) = 0,$$

so that  $z(x,y)$  is such that

$$z^2 - 2x^2 = u(y^2 - e^z)$$

with some  $u()$ . For  $x=y$ ,  $z(x,x)=0$ , we have

$$-2x^2 = u(x^2 - 1)$$

simplifying

$$u(t) = -2t - 2.$$

Therefore,

$$z^2 - 2x^2 = -2y^2 + 2e^z - 2$$

or

$$e^z - \frac{1}{2}z^2 = y^2 - x^2 + 1.$$

The left-hand side increases monotonically with  $z$  from  $-\infty$  to  $+\infty$  and therefore there is a unique  $z$  value for each  $x,y$  pair.

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[2] We eliminate  $u'$  (= derivative with respect to the argument) from

$$\frac{\partial}{\partial x} z = (z-1)(1+yu'(xy)),$$

$$\frac{\partial}{\partial y} z = (z-1) \times u'(xy)$$

to get

$$\boxed{\left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right) z = xz - x}$$

[3] The constraint says  $\int_0^a dx \times \delta y = 0$ ,

and the variation of the functional gives

$$\delta \int_0^a dx \times y'(x)^2 = 2 \int_0^a dx \times y' \delta y'$$

$$= 2xy' \delta y \Big|_{x=0}^a - 2 \int_0^a dx \delta y \frac{d}{dx}(xy')$$

$\underbrace{\hspace{10em}}_{=0}$  because  $y'(0)=0$   
 and  $\delta y(a)=0$

$$= -2 \int_0^a dx \delta y \frac{d}{dx}(xy'),$$

so that

$$\frac{d}{dx}(xy') = -\lambda x$$

with a Lagrange multiplier  $\lambda$ .

Question ... 3/5

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The solution of this differential equation is

$$y(x) = \left(1 + \frac{1}{4}\lambda\right) a^2 - \frac{1}{4}\lambda x^2$$

where  $y'(0) = 0$  and  $y(a) = a^2$  are taken into account. Now,

$$\begin{aligned} \int_0^a dx \times y(x) &= \left(1 + \frac{1}{4}\lambda\right) \frac{1}{2} a^4 - \frac{1}{16} \lambda a^4 \\ &= \left(\frac{1}{2} + \frac{1}{16}\lambda\right) a^4 \stackrel{!}{=} a^4, \end{aligned}$$

which establishes  $\lambda = 8$  and

$$y(x) = 3a^2 - 2x^2.$$

The smallest value of the functional is then

$$\int_0^a dx \times y'(x)^2 = \int_0^a dx \ 16x^3 = \underline{\underline{4a^4}}.$$

[4] We have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} (m\dot{y} - mxy) = m\dot{x}\dot{y} - m\omega^2 y = \frac{\partial L}{\partial x},$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = \frac{d}{dt} (m\dot{x} + mxy) = -m\dot{x}\dot{y} - m\omega^2 x = \frac{\partial L}{\partial y},$$

or

$$\begin{aligned} \ddot{x} &= -2\dot{x}\dot{y} - \omega^2 x, \\ \ddot{y} &= 2\dot{x}\dot{y} - \omega^2 y. \end{aligned}$$

