PC 3274, AY 2006/07 - I, Test 1

Question ......1/5........

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$\boxed{1}$ By the usual procedure we get

$$x e^{z} dx = yz\, dy,$$

$$2xy\, dy = x e^{z} dz, \qquad \text{or: } d(y^2 - e^{z}) = 0,$$

$$yz\, dz = 2xy\, dx, \qquad \text{or: } d(z^2 - 2x^2) = 0,$$

so that $z(x,y)$ is such that

$$z^2 - 2x^2 = u(y^2 - e^{z})$$

with some $u(\,)$. For $x=y$, $z(x,x)=0$, we have

$$-2x^2 = u(x^2 - 1)$$

implying

$$u(t) = -2t - 2 .$$

Therefore,

$$z^2 - 2x^2 = -2y^2 + 2e^{z} - 2$$

or

$$\boxed{e^{z} - \tfrac{1}{2} z^2 = y^2 - x^2 + 1} .$$

The left-hand side increases monotonically with z from $-\infty$ to $+\infty$ and therefore there is a unique z value for each x, y pair.

Question 2/5

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2 We eliminate u' (= derivative with respect to the argument) from

$$\frac{\partial}{\partial x} Z = (Z-1)\,(1 + y\,u'(xy)),$$

$$\frac{\partial}{\partial y} Z = (Z-1)\,x\,u'(xy)$$

to get

$$\boxed{\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) Z = xZ - x}\,.$$

3 The constraint says $\int_0^a dx\; x\,\delta y = 0$,

and the variation of the functional gives

$$\delta\int_0^a dx\; x\; y'(x)^2 = 2\int_0^a dx\; x\, y'\,\delta y'$$

$$= \underbrace{2xy'\delta y\Big|_{x=0}^{a}}_{=0 \text{ because } y'(0)=0 \text{ and } \delta y(a)=0} - 2\int_0^a dx\;\delta y\,\frac{d}{dx}(xy')$$

$$= -2\int_0^a dx\;\delta y\,\frac{d}{dx}(xy'),$$

so that

$$\frac{d}{dx}(xy') = -\lambda x$$

with a Lagrange multiplier λ.

Question 3/5

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The solution of this differential equation is

$$y(x) = (1+\tfrac{1}{4}\lambda)\, a^2 - \tfrac{1}{4}\lambda x^2$$

where $y'(0)=0$ and $y(a)=a^2$ are taken into account. Now,

$$\int_0^a dx\ x\, y(x) = (1+\tfrac{1}{4}\lambda)\tfrac{1}{2}a^4 - \tfrac{1}{16}\lambda a^4$$

$$= (\tfrac{1}{2}+\tfrac{1}{16}\lambda)a^4 \doteq a^4,$$

which establishes $\lambda = 8$ and

$$y(x) = 3a^2 - 2x^2 .$$

The smallest value of the functional is then

$$\int_0^a dx\ x\ y'(x)^2 = \int_0^a dx\ 16x^3 = \underline{\underline{4a^4}}.$$

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[4] We have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}(m\dot{y} - m\gamma y) = m\gamma\dot{y} - m\omega^2 y = \frac{\partial L}{\partial x},$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}(m\dot{x} + m\gamma x) = -m\gamma\dot{x} - m\omega^2 x = \frac{\partial L}{\partial y},$$

or

$$\boxed{\begin{aligned}\ddot{x} &= -2\gamma\dot{x} - \omega^2 x,\\ \ddot{y} &= 2\gamma\dot{y} - \omega^2 y.\end{aligned}}$$

Question 4/5

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The equation for $x(t)$ is the equation of motion of a damped harmonic oscillator.

5 With

$$p_x = m\dot{y} - m\gamma y, \quad p_y = m\dot{x} + m\gamma x$$

we have $m\dot{x} = p_y - m\gamma x$, $m\dot{y} = p_x + m\gamma y$,

and

$$H = p_x \dot{x} + p_y \dot{y} - L$$

with $L = (m\dot{x} + m\gamma x)\dot{y} + (m\dot{y} - m\gamma y)\dot{x} - m\dot{x}\dot{y}$
$\quad - m\omega^2 xy$

$$= p_y \dot{y} + p_x \dot{x} - m\dot{x}\dot{y} - m\omega^2 xy$$

so that $H = m\dot{x}\dot{y} + m\omega^2 xy$

$$= \frac{1}{m}(p_y - m\gamma x)(p_x + m\gamma y) + m\omega^2 xy$$

or, finally,

$$\boxed{H = \frac{1}{m} p_x p_y - \gamma x p_x + \gamma y p_y + m(\omega^2 - \gamma^2) xy.}$$

The Hamilton equations of motion are

$$\frac{d}{dt} x = \frac{\partial H}{\partial p_x} = \frac{1}{m} p_y - \gamma x,$$

$$\frac{d}{dt} y = \frac{\partial H}{\partial p_y} = \frac{1}{m} p_x + \gamma y,$$

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as well as $\frac{d}{dt} p_x = -\frac{\partial H}{\partial x} = \gamma p_x - m(\omega^2 - \gamma^2) y,$

$$\frac{d}{dt} p_y = -\frac{\partial H}{\partial y} = -\gamma p_y - m(\omega^2 - \gamma^2) x,$$

which are compactly summarized in

$$\frac{d}{dt}\begin{pmatrix} m\gamma x \\ m\gamma y \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} -\gamma & 0 & 0 & \gamma \\ 0 & \gamma & \gamma & 0 \\ 0 & \gamma - \omega^2/\gamma & \gamma & 0 \\ \gamma - \omega^2/\gamma & 0 & 0 & -\gamma \end{pmatrix}\begin{pmatrix} m\gamma x \\ m\gamma y \\ p_x \\ p_y \end{pmatrix}.$$