Problem 1 (25 marks)
Function $\mathsf{z}(x,y)$ obeys the qlPDE

$$\Big[y\mathsf{z}\frac{\partial}{\partial x}+x\mathrm{e}^{\mathsf{z}}\frac{\partial}{\partial y}\Big]\mathsf{z}=2xy\,.$$

Determine the solution of this qlPDE for $\mathsf{z}(x,x)=0$.
Note: You will not be able to state $\mathsf{z}(x,y)$ explicitly, but you can establish an algebraic equation obeyed by $\mathsf{z}(x,y)$ which one could solve numerically to determine $\mathsf{z}(x,y)$ for any given (x,y) pair. Show that the algebraic equation for $z(x,y)$ indeed has a unique solution for each (x,y) pair.

Problem 2 (15 marks)
The general solution of a certain qlPDE for $\mathsf{z}(x,y)$ is of the form

$$\mathsf{z}(x,y)=1+\mathrm{e}^{x+u(xy)}\quad\text{with arbitrary } u(\;)\,.$$

State this qlPDE.

Problem 3 (25 marks)
For given $a>0$, what is the smallest value that you can get for

$$\int_0^a \mathrm{d}x\,x\left[\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right]^2$$

if the permissible $y(x)$ are restricted by

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0)=0\,,\qquad y(a)=a^2\qquad\text{and}\qquad\int_0^a \mathrm{d}x\,xy(x)=a^4\,?$$

Problem 4 (15 marks)
Mass m is moving in the x,y plane whereby the Lagrange function

$$L(x,y,\dot{x},\dot{y})=m\dot{x}\dot{y}+m\gamma(x\dot{y}-\dot{x}y)-m\omega^2xy$$

applies, where γ and ω are positive time-independent parameters. State the Euler–Lagrange equations of motion in the form $\ddot{x}=\dots\,,\ \ddot{y}=\cdots$. What is the physical situation described by the equation for $x(t)$?

Problem 5 (20 marks)
Now derive the Hamilton function $H(x,p_x,y,p_y)$ that corresponds to the Lagrange function in Problem 4. Then state the Hamilton equations of motion in matrix form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}m\gamma x\\ m\gamma y\\ p_x\\ p_y\end{pmatrix}=\begin{pmatrix} \\ 4\times4\text{ matrix}\\ \\ \end{pmatrix}\begin{pmatrix}m\gamma x\\ m\gamma y\\ p_x\\ p_y\end{pmatrix}\,.$$