AY 2006/07, sem I, PC 3274, Test 2

Question 1/4

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1 Matrix multiplication is associative; the identity matrix $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is the neutral element, clearly an element of G; if M_1 and M_2 are in G, then also M_1M_2 because

$$(M_1M_2)^+\begin{pmatrix}0&1\\1&0\end{pmatrix}(M_1M_2)$$

$$= M_2^+\underbrace{M_1^+\begin{pmatrix}0&1\\1&0\end{pmatrix}M_1}_{=\begin{pmatrix}0&1\\1&0\end{pmatrix}}M_2 = M_2^+\begin{pmatrix}0&1\\1&0\end{pmatrix}M_2 = \begin{pmatrix}0&1\\1&0\end{pmatrix};$$

finally, we verify that $M^{-1}\in G$:

$$M^{-1^+}\begin{pmatrix}0&1\\1&0\end{pmatrix}M^{-1} = M^{-1^+}M^+\begin{pmatrix}0&1\\1&0\end{pmatrix}MM^{-1}$$

$$= \begin{pmatrix}0&1\\1&0\end{pmatrix}, \text{ indeed.}$$

If $\det\{M\}=1$, then $\det\{M^+\}=1$, and

$$\det\left\{M^+\begin{pmatrix}0&1\\1&0\end{pmatrix}M\right\} = \det\{M^+\}\det\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}\det\{M\}$$

$$= -1 = \det\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$$

and $\det\{M\}=1$ for $M=M_1M_2$ if $\det\{M_1\}=\det\{M_2\}=1$. It follows that G_+ is a subgroup of G.

$$G \ni \begin{pmatrix}0&1\\1&0\end{pmatrix} \notin G_+,$$

as one sees quickly.

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$\boxed{2}$ $M(a,b,c,d) = M(a_1, b_1, c_1, d_1)M(a_2, b_2, c_2, d_2)$

for $a = a_1 a_2 - b_1 c_2$,

$b = b_1 d_2 + a_1 b_2$,

$c = c_1 a_2 + d_1 c_2$,

$d = d_1 d_2 - c_1 b_2$.

(a) When $b_1 = c_1$, $a_1 = d_1$, $b_2 = c_2$, $a_2 = d_2$, then

$$a = a_1 a_2 - b_1 b_2 = d,$$
$$b = b_1 a_2 + a_1 b_2 = c,$$

so that we have a subgroup, and since the expressions for a and b are invariant under the interchange $1 \leftrightarrow 2$, the subgroup is Abelian.

(b) When $b_1 = 0$ and $b_2 = 0$, then also $b = 0$, so that we have a subgroup, but it is not Abelian because $c = c_1 a_2 + d_1 c_2$ differs from $c_2 a_1 + d_2 c_1$ as a rule.

(c) When $c_1 = 0$ and $c_2 = 0$, then also $c = 0$, so that this also gives a subgroup, but again it is not Abelian.

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(d) $M^+ = M$ requires $c = -b$. Now when $c_1 = -b_1$ and $c_2 = -b_2$, then

$$b = b_1 d_2 + a_1 b_2$$

and $c = -b_1 a_2 - d_1 b_2$

so that $b \neq -c$ as a rule and we do <u>not</u> have <u>a subgroup</u>.

$\boxed{3}$ Recall $\int_0^\infty dt\, e^{-st} t^n = \frac{n!}{s^{n+1}}$ and use the convolution theorem to establish

$$\int_0^\infty dt\, e^{-st} \int_0^t d\tau\, (t-\tau)^m \tau^n = \frac{m!}{s^{m+1}} \frac{n!}{s^{n+1}}$$

$$= \frac{m!\,n!}{(m+n+1)!} \frac{(m+n+1)!}{s^{(m+n+1)+1}}$$

$$= \int_0^\infty dt\, e^{-st} \frac{m!\,n!}{(m+n+1)!} t^{m+n+1},$$

so that

$$\int_0^t d\tau\, (t-\tau)^m \tau^n = \frac{m!\,n!}{(m+n+1)!} t^{m+n+1}$$

follows.

Question. 4/4

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$\boxed{4}$ We have $T_{k+1} = T_k (E + \lambda \epsilon_k)$, so that

$$\epsilon_{k+1} = (E + \lambda^* \epsilon_k) \underbrace{T_{k+1} S T_k}_{= E + \epsilon_k} (E + \lambda \epsilon_k) - E$$

$$= (1 + \lambda + \lambda^*) \epsilon_k + (\lambda + \lambda^* + |\lambda|^2) \epsilon_k^2 + |\lambda|^2 \epsilon_k^3$$

or $C_1 = 1 + \lambda + \lambda^*$, $C_2 = \lambda + \lambda^* + |\lambda|^2$,

$C_3 = |\lambda|^2$.

For $C_1 = 0$, we need $\lambda + \lambda^* = -1$, and we can have $C_2 = 0$ if $|\lambda|^2 = 1$ in addition, so that

$$\lambda = -\tfrac{1}{2} + i \tfrac{1}{2}\sqrt{3} \quad \text{or} \quad \lambda = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}$$

is optimal, because then $C_1 = 0$, $C_2 = 0$, and $C_3 = 1$, giving

$\epsilon_{k+1} = \epsilon_k^3$: cubic convergence, which is very fast. Here

$$\epsilon_0 = S - E, \quad \epsilon_1 = (S-E)^3, \quad \epsilon_2 = (S-E)^9,$$

and so forth.

We have $S^{-1} = T T^+$.