PC3274, AY 2006/07, Sem I, Exam

Question 1/6

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$\boxed{1}$ The $y(x)$ that minimizes the integral over $y'(x)^2$ obeys the differential equation

$$y''(x) = -6\lambda|x|,$$

where λ is the Lagrange multiplier for the constraint and the factor of -6 is for convenience. The solution that takes $y(\pm a)=0$ into account is

$$y(x) = \lambda\left(a^3 - |x|^3\right),$$

with the value of λ determined by

$$a^3 = \int_{-a}^{a} dx\ |x|\ y(x) = 2\lambda\int_0^a dx\left(a^3x - x^4\right)$$

$$= \frac{3}{5}\lambda a^5, \quad \text{so that } \lambda = \frac{5}{3a^2}.$$

The minimal value is, therefore,

$$\int_{-a}^{a} dx\ \left(-3\lambda x|x|\right)^2 = 18\lambda^2\int_0^a dx\ x^4$$

$$= \frac{18}{5}\lambda^2 a^5 = \frac{18}{5}\left(\frac{5}{3a^2}\right)^2 a^5 = \underline{\underline{10a}}.$$

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2 (a) Closure is obvious (if not: see below, page 4).
Neutral element: $e = (1, 0)$.
Inverse element: $g^{-1} = (a^*, -b)$.

Associativity For $g_1(g_2 g_3) = (g_1 g_2) g_3$ we need

$$a_1(a_2a_3 + b_2^* b_3) + b_1^*(b_2 a_3 + a_2^* b_3)$$

$$= (a_1 a_2 + b_1^* b_2) a_3 + (b_1 a_2 + a_1^* b_2)^* b_3$$

and

$$b_1(a_2 a_3 + b_2^* b_3) + a_1^*(b_2 a_3 + a_2^* b_3)$$

$$= (b_1 a_2 + a_1^* b_2) a_3 + (a_1 a_2 + b_1^* b_2)^* b_3 ;$$

both are identities indeed.

(b) For $a = a_1 a_2 + b_1^* b_2$, $b = b_1 a_2 + a_1^* b_2$ we have

(i, ii) $\mathrm{Im}(a \mp b) = \mathrm{Im}\left((a_1 \mp b_1) a_2 \mp (a_1 \mp b_1)^* b_2\right)$

$$= \mathrm{Im}(a_1 \mp b_1)\,\mathrm{Re}(a_2 \pm b_2) + \mathrm{Re}(a_1 \mp b_1)\,\mathrm{Im}(a_2 \mp b_2).$$

Therefore, if (i) $\mathrm{Im}\, a_1 = \mathrm{Im}\, b_1$ and $\mathrm{Im}\, a_2 = \mathrm{Im}\, b_2$ then also $\mathrm{Im}\, a = \mathrm{Im}\, b$;

and if (ii) $\mathrm{Im}\, a_1 = -\mathrm{Im}\, b_1$ and $\mathrm{Im}\, a_2 = -\mathrm{Im}\, b_2$ then also $\mathrm{Im}\, a = -\mathrm{Im}\, b$.

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Accordingly there is closure under (i) and (ii). Further $\operatorname{Im} a = \operatorname{Im} b = 0$ for the neutral element, so that $\operatorname{Im} a = \pm \operatorname{Im} b$ for e; and, finally, if $\operatorname{Im} a = \pm \operatorname{Im} b$ for $g = (a, b)$, then also for $g^{-1} = (a^*, -b)$.
Conclusion: (i) and (ii) define subgroups.

(iii) For $g_1 = (-i\sqrt{2}, 1)$, $g_2 = (i\sqrt{2}, 1)$,

we have $g_1 g_2 = (3, i\sqrt{8})$, so that

$\operatorname{Im} b \neq 0$ although $\operatorname{Im} b_1 = \operatorname{Im} b_2 = 0$.
This example therefore demonstrates the lack of closure. Conclusion: (iii) does not define a subgroup.

(iv) $b_1 = b_2 = 0$ imply $b = 0$, so that restriction (iv) defines a subgroup.

(C) The subgroup for (iv) is abelian, those for (i) and (ii) are not.

Case for (iv): $g_1 = (a_1, 0)$, $g_2 = (a_2, 0)$ give $g_1 g_2 = (a_1 a_2, 0) = g_2 g_1$.

Case for (i) take $g_1 = (1+i, i)$, $g_2 = (\sqrt{2}, 1)$ to show that $g_1 g_2 \neq g_2 g_1$; and similarly for (ii).

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Returning to (a) closure: We would need to verify that

$$|a_1 a_2 + b_1^* b_2|^2 - |b_1 a_2 + a_1^* b_2|^2 = 1$$

if $|a_1|^2 - |b_1|^2 = 1$ and $|a_2|^2 - |b_2|^2 = 1$.

See: $|a_1|^2|a_2|^2 + |b_1|^2|b_2|^2 - |b_1|^2|a_2|^2 - |a_1|^2|b_2|^2$

$$+ \underbrace{a_1^* a_2^* b_1^* b_2 + a_1 a_2 b_1 b_2^* - b_1^* a_2^* a_1^* b_2 - b_1 a_2 a_1 b_2^*}_{=0}$$

$$= (|a_1|^2 - |b_1|^2)(|a_2|^2 - |b_2|^2) = 1, \text{ indeed.}$$

$\boxed{3}$ We have $f(t + T/2) = -f(t)$ and $f(t) = 1$ for $0 < t < T/2$, so that

$$F(s) = \int_0^\infty dt\, e^{-st} f(t) = \sum_{k=0}^\infty \int_{kT/2}^{(k+1)T/2} dt\, e^{-st} f(t)$$

$$= \sum_{k=0}^\infty \int_0^{T/2} dt\, e^{-s(t + kT/2)} \underbrace{f(t + kT/2)}_{=(-1)^k}$$

$$= \sum_{k=0}^\infty (-1)^k e^{-ksT/2} \int_0^{T/2} dt\, e^{-st}$$

$$= \frac{1}{1 + e^{-sT/2}} \quad \frac{1 - e^{-sT/2}}{s}$$

$$= \frac{1}{s} \tanh(sT/4).$$

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$\boxed{4}$ (a) We have

$$\int_C \frac{dz}{2\pi i z} z^n e^{\frac{1}{2}t(z-1/z)}$$

$$= \begin{cases} \int_C \frac{dz}{2\pi i} \sum_{m=-\infty}^{\infty} z^{m+n-1} b_m(t) = b_{-n}(t) \\ \int_C \frac{dz}{2\pi i} \sum_{m=-\infty}^{\infty} z^{-m+n-1} (-1)^m b_m(t) = (-1)^n b_n(t) \end{cases}$$

since $\int_C \frac{dz}{2\pi i} z^{k-1} = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{if } k=\pm1, \pm2, \pm3, \ldots \end{cases}$

(b) We take $n = 0, 1, 2, \ldots$ and get

$$(-1)^n B_n(s) = B_{-n}(s)$$

$$= \int_0^\infty dt\, e^{-st} \int_C \frac{dz}{2\pi i z} z^n e^{\frac{1}{2}t(z-1/z)}$$

$$= \int_C \frac{dz}{2\pi i z} z^n \frac{1}{s - \frac{1}{2}(z - 1/z)}$$

$$= \int_C \frac{dz}{2\pi i} \frac{2z^n}{1+2sz-z^2}$$

$$= \int_C \frac{dz}{2\pi i} \frac{2z^n}{(z-z_1)(z_2-z)}$$

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with $z_1 z_2 = -1$ and $z_1 + z_2 = 2s$

or $z_1 = s + \sqrt{1+s^2}$

and $z_2 = s - \sqrt{1+s^2}$.

Since $z_1 > 1$ and $-1 < z_2 < 0$ for $s > 0$, the pole at z_1 is outside the unit circle whereas the pole at z_2 is inside. In terms of the residue at $z = z_2$, we thus get

$$(-1)^n B_n(s) = B_{-n}(s)$$

$$= \frac{2 z_2^n}{z_1 - z_2} = \frac{(s - \sqrt{1+s^2})^n}{\sqrt{1+s^2}} ,$$

so that

$$B_n(s) = (-1)^n B_{-n}(s) = \frac{(\sqrt{1+s^2} - s)^n}{\sqrt{1+s^2}}$$

for $n = 0, 1, 2, \ldots$.

(c) We have

$$\int_0^\infty dt\, b_0(t) = B_0(0) = \underline{\underline{1}}$$

and

$$\int_0^\infty dt\, \frac{b_1(t)}{t} = \int_0^\infty ds\, B_1(s)$$

$$= \int_0^\infty ds \left(1 - \frac{s}{\sqrt{1+s^2}}\right) = \left(s - \sqrt{1+s^2}\right)\Big|_{s=0}^{\infty} = \underline{\underline{1}} .$$