

Write answers on this side of the paper only.

Do not write on either margin

□ (a) Obviously

$$\delta(A, a_1=1) = \frac{1}{3} (A-2)(A-4),$$

$$\delta(A, a_2=2) = -\frac{1}{2} (A-1)(A-4),$$

$$\delta(A, a_3=4) = \frac{1}{6} (A-1)(A-2),$$

where the  $A-a_j$  factors ensure the vanishing of the RHSs for the other eigenvalues, and the prefactors are such that the RHSs = 1 for the respective eigenvalue.

$$\begin{aligned} \text{(b) Since } \log_2 A &= \sum_j |a_j\rangle \log_2 a_j \langle a_j| \\ &= \sum_j \delta(A, a_j) \log_2 a_j \\ &= \delta(A, 2) + 2 \delta(A, 4) \end{aligned}$$

we get

$$\log_2 A = \frac{1}{6} (A-1)(8-A).$$

As a check one verifies easily that both sides give the same values for  $A = 1, 2, \text{ or } 4$ .

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[2] (a) We need to show that  $|\langle u|v\rangle|^2 = \frac{1}{N}$  for any eigenvector  $|u\rangle$  of  $U$  and any eigenvector  $|v\rangle$  of  $V$ . So,

$$\begin{aligned} |\langle u|v\rangle|^2 &= \text{tr}\{|u\rangle\langle u|v\rangle\langle v|\} \\ &= \text{tr}\left\{\frac{1}{N}\sum_m (U|u\rangle)^m \frac{1}{N}\sum_n (V|v\rangle)^n\right\} \\ &\stackrel{\substack{(1.2.13) \\ (1.2.22)}}{\uparrow} = \frac{1}{N^2} \sum_{m,n} u^{-m} v^{-n} \text{tr}\{U^m V^n\} \end{aligned}$$

and the only term contributing to the sum has  $U^m = 1$  and  $V^n = 1$  and therefore also  $u^m = 1$  and  $v^n = 1$ , so that

$$|\langle u|v\rangle|^2 = \frac{1}{N^2} N = \frac{1}{N}, \text{ indeed.}$$

(b)  $U^j V$  is unitary, because it is a product of unitary operators.

$$(U^j V)^N = e^{i\phi_j} U^{jN} V^N = e^{i\phi_j}$$

with some phase factor, as implied by (1.2.36).

(c) Two cases to consider (i)  $j=0 < k$ ,  
(ii)  $0 < j < k$ .

(i) For  $j=0 < k$  we have

$$\begin{aligned} W_j^m W_k^n &= U^m (U^k V)^n \\ &= U^{m+kn} V^n \times (\text{some phase factor}) \end{aligned}$$

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and we get  $\text{tr}\{W_j^m W_k^n\} \neq 0$  only if  $V^n = 1$   
and  $U^{m+kn} = 1$ . But if  $V^n = 1$ , then  
also  $U^{kn} = (U^n)^j = 1$ , so that  $U^m = 1$   
is required. Together we have

$$\text{tr}\{W_j^m W_k^n\} = 0 \text{ unless } W_j^m = 1 \text{ and } W_k^n = 1$$

for  $0 = j < k$  (up to the phase factors of part (b)).

(ii) Now we have

$$W_j^m W_k^n = (U^j V)^m (U^k V)^n$$

$$= (\text{phase factor}) U^{j(m+kn)} V^{m+n}$$

so that  $\text{tr}\{W_j^m W_k^n\} = 0$  unless

$m+n$  is a multiple of  $N$  and

$j(m+kn)$  is a multiple of  $N$ . But

since  $N$  is prime this is only possible if both  $m$  and  $n$  are multiples of  $N$ , so that we need

$$W_j^m = 1 \text{ and } W_k^n = 1$$

(up to the phase factors of part (b))

also here for  $0 < j < k \leq N$ .

Together we have  $\text{tr}\{W_j^m W_k^n\} = 0$  unless  $W_j^m$  and  $W_k^n$  are multiples of the identity, which, in view of (a), proves the case.

