AY2006/07, Sem I, PC4130, Test 1

Question 1/6

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$\boxed{1}$ (a) Obviously

$$\delta(A, a_1=1) = \frac{1}{3}(A-2)(A-4),$$

$$\delta(A, a_2=2) = -\frac{1}{2}(A-1)(A-4),$$

$$\delta(A, a_3=4) = \frac{1}{6}(A-1)(A-2),$$

where the $A-a_j$ factors ensure the vanishing of the RHSs for the other eigenvalues, and the prefactors are such that the RHSs $=1$ for the respective eigenvalue.

(b) Since
$$\begin{aligned}\log_2 A &= \sum_j |a_j\rangle \log_2 a_j \langle a_j| \\ &= \sum_j \delta(A, a_j) \log_2 a_j \\ &= \delta(A,2) + 2\,\delta(A,4)\end{aligned}$$

we get

$$\log_2 A = \frac{1}{6}(A-1)(8-A).$$

As a check one verifies easily that both sides give the same values for $A = 1, 2,$ or 4.

Question 2/6

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$\boxed{2}$ (a) We need to show that $|\langle u|v\rangle|^2 = \frac{1}{N}$ for any eigenbra $\langle u|$ of U and any eigenket $|v\rangle$ of V. So,

$$|\langle u|v\rangle|^2 = \operatorname{tr}\{|u\rangle\langle u|v\rangle\langle v|\}$$

$$\overset{\left(\substack{(1.2.13)\\(1.2.22)}\right)}{=} \operatorname{tr}\left\{\frac{1}{N}\sum_m (U/u)^m \; \frac{1}{N}\sum_n (V/v)^n\right\}$$

$$= \frac{1}{N^2}\sum_{m,n} u^{-m} v^{-n} \operatorname{tr}\{U^m V^n\}$$

and the only term contributing to the sum has $U^m = 1$ and $V^n = 1$ and therefore also $u^m = 1$ and $v^n = 1$, so that

$$|\langle u|v\rangle|^2 = \frac{1}{N^2} N = \frac{1}{N} \text{, indeed.}$$

(b) $U^j V$ is unitary, because it is a product of unitary operators.

$$(U^j V)^N = e^{i\phi_j} \; U^{jN} V^N = e^{i\phi_j}$$

with some phase factor, as implied by (1.2.36).

(c) Two cases to consider (i) $j = 0 < k$,
(ii) $0 < j < k$.

(i) For $j = 0 < k$ we have

$$W_j^m W_k^n = U^m (U^k V)^n$$

$$= U^{m+kn} V^n \times (\text{some phase factor})$$

Question ...3/6....

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and we get $\mathrm{tr}\{W_j^m W_k^n\} \neq 0$ only if $V^n = 1$ and $U^{jm+kn} = 1$. But if $V^n = 1$, then also $U^{kn} = (U^n)^j = 1$, so that $U^m = 1$ is required. Together we have

$$\mathrm{tr}\{W_j^m W_k^n\} = 0 \quad \text{unless } W_j^m = 1 \text{ and } W_k^n = 1$$

for $0 = j < k$ (up to the phase factors of part (b)).

(ii) Now we have

$$W_j^m W_k^n = (U^j V)^m (U^k V)^n$$

$$= (\text{phase factor})\, U^{jm+kn} V^{m+n}$$

so that $\mathrm{tr}\{W_j^m W_k^n\} = 0$ unless $m+n$ is a multiple of N and $jm+kn$ is a multiple of N. But <u>since N is prime</u> this is only possible if both m and n are multiples of N, so that we need

$$W_j^m = 1 \text{ and } W_k^n = 1$$

(up to the phase factors of part (b)) also here for $0 < j < k \leq N$.

Together we have $\mathrm{tr}\{W_j^m W_k^n\} = 0$ unless W_j^m and W_k^n are multiples of the identity which, in view of (a), proves the case.

(d) Here $U^4=1$, $V^4=1$ and

$W_0 = U$, $W_1 = UV$, $W_2 = U^2V$, $W_3 = U^3V$, $W_4 = V$.

Take for example

$$\mathrm{tr}\{W_2^2 W_4^2\} = \mathrm{tr}\{U^2VU^2V^3\}$$

$$= \mathrm{tr}\{\underbrace{e^{i\pi}}_{=-1} \underbrace{U^4V^4}_{=1}\} = -N$$

so that $\mathrm{tr}\{W_j^m W_k^n\} \neq 0$ although W_j^m and W_k^n are <u>not</u> multiples of the identity, namely

$$W_2^2 = -V^2, \quad W_4^2 = V^2.$$

It follows that W_2, W_4 are <u>not</u> a <u>pair of complementary operators.</u>

$\boxed{3}$ (a) With $\frac{\partial}{\partial\lambda} H = -\delta(X(t)-x_0) = -|x_0,t\rangle\langle x_0,t|$,

we get

$$\delta\langle x,t_1|x',t_2\rangle = \frac{i}{\hbar}\int_{t_2}^{t_1} dt\left(-\frac{\partial}{\partial\lambda}H\right)$$

$$= \frac{i}{\hbar}\int_{t_2}^{t_1} dt\,\langle x,t_1|x_0,t\rangle\langle x_0,t|x',t_2\rangle.$$

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(b) For $\lambda = 0$, we have the time transformation function of (1.3.18), so that, for $x = x' = x_0$,

$$\frac{\partial}{\partial\lambda}\langle x_0, t_1 | x_0, t_2\rangle\Big|_{\lambda=0} = \frac{i}{\hbar}\int_{t_2}^{t_1} dt \sqrt{\frac{M}{i2\pi\hbar(t_1-t)}}\sqrt{\frac{M}{i2\pi\hbar(t-t_2)}}$$

$$= \frac{i}{\hbar}\,\frac{M}{i2\pi\hbar}\int_{t_2}^{t_1}\frac{dt}{\sqrt{(t_1-t)(t-t_2)}}.$$

Substitute $t = \frac{1}{2}(t_1+t_2) - \frac{1}{2}(t_1-t_2)\cos\vartheta$ with $0 \le \vartheta \le \pi$ to cover $t_2 \le t \le t_1$ and get

$$\frac{\partial}{\partial\lambda}\langle x_0, t_1 | x_0, t_2\rangle\Big|_{\lambda=0} = \frac{M}{2\pi\hbar^2}\int_0^{\pi} d\vartheta = \underline{\frac{M}{2\hbar^2}}.$$

$\boxed{4}$ (a) $\delta_1\,\delta_2 \det\{A\} = \delta_1\left(\det\{A\}\,\mathrm{tr}\{A^{-1}\delta_2 A\}\right)$

$$= \left(\delta_1 \det\{A\}\right)\mathrm{tr}\{A^{-1}\delta_2 A\}$$

$$+ \det\{A\}\,\mathrm{tr}\{\underbrace{(\delta_1 A^{-1})}_{= -A^{-1}\delta_1 A\, A^{-1}}\delta_2 A + A^{-1}\delta_1\delta_2 A\}$$

$$= \det\{A\}\left[\mathrm{tr}\{A^{-1}\delta_1 A\}\,\mathrm{tr}\{A^{-1}\delta_2 A\} + \mathrm{tr}\{A^{-1}\delta_1\delta_2 A - A^{-1}\delta_1 A\, A^{-1}\delta_2 A\}\right]$$

which, because of the cyclic property of the trace, is invariant under the interchange $1 \leftrightarrow 2$, as it should be.

Question ...6/6.......

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(b) We have $\delta(AB) = \delta A\, B + A\,\delta B$, so that

$$\delta \det\{AB\} = \det\{AB\}\, \mathrm{tr}\{B^{-1}A^{-1}(\delta A B + A \delta B)\}$$
$$= \det\{AB\}\, \mathrm{tr}\{A^{-1}\delta A + B^{-1}\delta B\},$$

or

$$\frac{\delta \det\{AB\}}{\det\{AB\}} = \frac{\delta \det\{A\}}{\det\{A\}} + \frac{\delta \det\{B\}}{\det\{B\}}$$

or

$$\delta \log \det\{AB\} = \delta \log \det\{A\} + \delta \log \det\{B\}$$
$$= \delta \log(\det\{A\} \det\{B\}).$$

It follows that $\det\{AB\} = \det\{A\}\det\{B\}$ up to a multiplicative constant, which however must equal 1, as we see for $B = 1$, $\det\{B\} = 1$.

(c) For $A = e^Z$, we have $\delta A = \int_0^1 dx\, e^{(1-x)Z}\,\delta Z\, e^{xZ}$

and $$\mathrm{tr}\{A^{-1}\delta A\} = \int_0^1 dx\, \underbrace{\mathrm{tr}\{e^{-Z}e^{(1-x)Z}\cdot \delta Z\, e^{xZ}\}}_{=\mathrm{tr}\{\delta Z\}}$$

$$= \mathrm{tr}\{\delta Z\} = \delta\, \mathrm{tr}\{Z\},$$

so that

$$\delta \det\{e^Z\} = \det\{e^Z\}\, \delta\, \mathrm{tr}\{Z\}$$

and $\det\{e^Z\} = e^{\mathrm{tr}\{Z\}}$ follows immediately.