Problem 1 (15=9+6 points)
Operator A has three different eigenvalues: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$.

(a) Write the projection operators $\big|a_j\big\rangle\big\langle a_j\big| = \delta(A, a_j)$ $(j = 1, 2, 3)$ as polynomials in A of the lowest possible degree.

(b) Write the operator function $f(A) = \log_2 A$ as a polynomial in A of the lowest possible degree.

Problem 2 (35=15+5+10+5 points)
Operators U and V are two cyclic unitary operators of period N for an N-dimensional quantum degree of freedom.

(a) We know from the lecture and the tutorials that, for all integer values of n and m,

$$\mathrm{tr}\big\{U^m V^n\big\} = \begin{cases} N & \text{if } U^m = 1 \text{ and } V^n = 1\,, \\ 0 & \text{otherwise,} \end{cases} \qquad (*)$$

<u>*if*</u> U and V are a pair of complementary observables. Now show the converse: *If* (*) holds, then U and V are a complementary pair. — Hint: Recall how to express a squared bracket in terms of a trace.

(b) Now consider the set of $N + 1$ operators defined by

$$W_0 = U\,,\ W_1 = UV\,,\ W_2 = U^2V\,,\ \ldots\,,\ W_{N-1} = U^{N-1}V\,,\ W_N = V\,,$$

where U and V are the usual pair of complementary unitary operators. For $j = 1, 2, \ldots, N - 1$, show that $W_j = U^jV$ is unitary and ${W_j}^N$ is a multiple of the identity.

(c) Then show that each pair W_j, W_k $(0 \leq j < k \leq N)$ is a complementary pair <u>*if*</u> N is prime.

(d) For $N = 4$, find a pair W_j, W_k that is *not* a complementary pair.

Problem 3 (20=10+10 points)
Mass M moves along the x axis whereby the Hamilton operator

$$H = \frac{1}{2M}P^2 - \lambda\,\delta(X - x_0) \quad \text{with constant } \lambda \text{ and } x_0$$

governs the evolution. The time transformation function $\langle x, t_1 | x', t_2 \rangle$ depends on the strength λ of the coupling to, and the location x_0 of, the delta-function potential $\delta\bigl(X(t) - x_0\bigr) = |x_0, t\rangle\langle x_0, t|$.

(a) Use the quantum action principle to express $\dfrac{\partial}{\partial\lambda}\langle x, t_1 | x', t_2 \rangle$ as an integral over the intermediate time t.

(b) Then recall the $\lambda = 0$ form of the time transformation function and determine the value of $\dfrac{\partial}{\partial\lambda}\langle x_0, t_1 | x_0, t_2 \rangle\Big|_{\lambda\,=\,0}$. — Hint: A parameterization that we used in lecture for an integral on page 40 of the notes could be useful.

Problem 4 (30=10+10+10 points)
For an operator A that has an inverse A^{-1}, one can define the determinant by the differential statement

$$\delta\det\{A\} = \det\{A\}\,\mathrm{tr}\left\{A^{-1}\,\delta A\right\} \qquad (**)$$

together with $\det\{1\} = 1$.

(a) Consistency requires that $\delta_1\delta_2\det\{A\} = \delta_2\delta_1\det\{A\}$ if δ_1 and δ_2 symbolize two *independent* infinitesimal variations. Verify that this is indeed correct. — Hint: You will need an expression for δA^{-1}.

(b) Use (**) to show that $\det\{AB\} = \det\{A\}\det\{B\}$.

(c) Use (**) to express $\det\left\{\mathrm{e}^{Z}\right\}$ in terms of $\mathrm{tr}\{Z\}$.