AY2006/07, Sem I, PC4130, 2nd Test

Question. . 1/5 . . . .

Write answers on this side of the paper only.

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[1] Since $\tilde{S}(T) = 1 - \frac{i}{\hbar}\int_0^T dt\, \underbrace{\overline{W^+ H_1 W}(t)}_{= W^+ \overline{H_1}(t) W} \tilde{S}(t)$

$$= 1 - \frac{i}{\hbar} W^+ \int_0^T dt\, \overline{H_1}(t)\, W \tilde{S}(t),$$

we have $W\tilde{S}(T)W^+ = 1 - \frac{i}{\hbar}\int_0^T dt\, \overline{H_1}(t)\, W\tilde{S}(t)W^+,$

so that $W\tilde{S}(T)W^+$ obeys the equation for $S(T)$. It follows that $S(T) = W\tilde{S}(T)W^+$, or $\tilde{S}(T) = W^+ S(T) W.$

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[2] The Hamilton operator is

$$H = \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2 X^2 - F(t)X$$

$$= \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2\left(X - \frac{F}{M\omega^2}\right)^2 - \frac{F^2}{2M\omega^2}$$

so that the <u>instantaneous</u> ground state is that of an oscillator centered at $x = F/(M\omega^2)$, and the instantaneous ground state wave function is the correspondingly displaced gaussian:

$$\psi_T = \frac{(2\pi)^{-1/4}}{\sqrt{\delta X}}\, e^{-\left(\frac{X - x_0}{2\delta X}\right)^2}$$

with $\delta x = \sqrt{\frac{\hbar}{2M\omega}}$ and $x_\infty = \frac{F_\infty}{M\omega^2}$.

The asked-for probability is, therefore, given by the <u>square</u> of

$$\int_{-\infty}^{\infty} dx \, \frac{(2\pi)^{-1/4}}{\sqrt{\delta x}} \, e^{-\left(\frac{x}{2\delta x}\right)^2} \, \frac{(2\pi)^{-1/4}}{\sqrt{\delta x}} \, e^{-\left(\frac{x-x_\infty}{2\delta x}\right)^2}$$

$x \to x + \frac{1}{2}x_\infty$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\delta x} \int_{-\infty}^{\infty} dx \; e^{-\left(\frac{1}{2\delta x}\right)^2 \left[\left(x+\frac{1}{2}x_\infty\right)^2 + \left(x - \frac{1}{2}x_\infty\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\delta x} \sqrt{2\pi}\, \delta x \; e^{-\left(\frac{1}{2\delta x}\right)^2 \frac{1}{2} x_\infty^2}$$

$$= e^{-\frac{1}{8}\left(x_\infty/\delta x\right)^2}$$

$$= e^{-\frac{F_\infty^2}{4\hbar M\omega^3}} \, .$$

$\boxed{3}$ The transition rate for one particular λ is

$$\gamma_\lambda = \frac{2\pi}{\hbar} \left| \langle 0 | \, a_\lambda \int d\lambda' \, \hbar \Omega_{\lambda'} \left(A^\dagger a_{\lambda'} + a_{\lambda'}^\dagger A\right) A^\dagger | 0 \rangle \right|^2$$

$$\times \, \delta(\hbar\omega - \hbar\omega_\lambda)$$

$$= \frac{2\pi}{\hbar} \left| \hbar \Omega_\lambda \right|^2 \delta(\hbar\omega - \hbar\omega_\lambda)$$

$$= 2\pi \left| \Omega_\lambda \right|^2 \delta(\omega - \omega_\lambda)$$

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with the density of states denoted by $\rho(\omega)$ and averaging over all ω_λ with $\omega_\lambda = \omega$, we get

$$\gamma = 2\pi \rho(\omega) \, \overline{|\Omega_\lambda|^2}^{(av)} .$$

$\boxed{4}$ First we need to express the state with $j=\frac{1}{2}, m=\frac{1}{2}$ in terms of the states with $j_1 = 1, j_2 = \frac{1}{2}$ and $m_1 + m_2 = \frac{1}{2}$. We begin with the state $j=\frac{3}{2}, m=\frac{3}{2}$:

$$|j=\tfrac{3}{2}, m=\tfrac{3}{2}\rangle = |m_1=1, m_2=\tfrac{1}{2}\rangle \equiv |1;\tfrac{1}{2}\rangle$$

and apply J_-:

$$|j=\tfrac{3}{2}, m=\tfrac{1}{2}\rangle \propto J_- |1;\tfrac{1}{2}\rangle$$

$$= |0;\tfrac{1}{2}\rangle \hbar \sqrt{(1+1)(1-1+1)}$$

$$+ |1;-\tfrac{1}{2}\rangle \hbar \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}$$

$$= \left(|0;\tfrac{1}{2}\rangle\sqrt{2} + |1;-\tfrac{1}{2}\rangle\right)\hbar .$$

The state $|j=\frac{1}{2}, m=\frac{1}{2}\rangle$ is orthogonal to this one and is a superposition of the same two kets. Properly normalized we have

$$|j=\tfrac{1}{2}, m=\tfrac{1}{2}\rangle = \left(|1;-\tfrac{1}{2}\rangle\sqrt{2} - |0;\tfrac{1}{2}\rangle\right)/\sqrt{3} .$$

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We can now read off the respective probability amplitudes and determine the asked for probabilities.

(a) These probabilities are

$\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$ for $m_1 = 1$,

$\left(-\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{3}$ for $m_1 = 0$,

0 for $m_1 = -1$.

(b) These probabilities are

$\left(-\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{3}$ for $m_2 = \frac{1}{2}$,

$\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}$ for $m_2 = -\frac{1}{2}$.

$\boxed{5}$ (a) Since this is obtained from $|j=\frac{3}{2}, m=\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$ by an application of $J_- = J_{1-} + J_{2-} + J_{3-}$, it must be the symmetric superposition

$$|j=\tfrac{3}{2}, m=\tfrac{1}{2}\rangle = \left(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right)/\sqrt{3}.$$

(b) All superposition orthogonal to $|j=\frac{3}{2}, m=\frac{1}{2}\rangle$ will have $|j=\frac{1}{2}, m=\frac{1}{2}\rangle$, the general case being

$$|\uparrow\uparrow\downarrow\rangle\alpha + |\uparrow\downarrow\uparrow\rangle\beta + |\downarrow\uparrow\uparrow\rangle\gamma$$

with $\alpha+\beta+\gamma=0$ and $|\alpha|^2+|\beta|^2+|\gamma|^2=1$.

Examples are

$$|\,1st\rangle = \left(|\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle\right)/\sqrt{2}$$

that is; $\alpha = 1/\sqrt{2}$, $\beta = 0$, $\gamma = -1/\sqrt{2}$

and then

$$|\,2nd\rangle = \left(|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle 2 + |\downarrow\uparrow\uparrow\rangle\right)/\sqrt{6}$$

that is : $\alpha = \frac{1}{\sqrt{6}}$, $\beta = \frac{-2}{\sqrt{6}}$, $\gamma = \frac{1}{\sqrt{6}}$.

There is no third state because we only have three kets available to form the pairwise orthogonal $m = \frac{1}{2}$ states.