Do not write on either margin

[1] Since $0 = [A, BB^{-1}] = B[A, B^{-1}] + [A, B]B^{-1}$ we have

$$[A, B^{-1}] = -B^{-1}[A, B]B^{-1}.$$

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[2] The oscillator will remain in the instantaneous ground state. In view of

$$H = \hbar\omega\left(A^{+} - \frac{\Omega^{*}}{\omega}\right)\left(A - \frac{\Omega}{\omega}\right) - \hbar\frac{|\Omega|^2}{\omega}$$

the instantaneous ground state is the ground state of an oscillator with $\langle A\rangle = \Omega/\omega$, $\langle A^{+}\rangle = \Omega^{*}/\omega$, so that

$$\rho = e^{-(A^{+} - \Omega^{*}/\omega);(A - \Omega/\omega)}$$

with $\Omega = \Omega(t)$ applies at intermediate times and

$$\rho(T) = e^{-(A^{+} - \Omega_{\infty}^{*}/\omega);(A - \Omega_{\infty}/\omega)}$$

is the statistical operator (= projector to the ground state) at time T.

(a) With $A|0\rangle = 0$ and $\langle 0|A^{+} = 0$, we have

$$\langle 0|\rho(T)|0\rangle = e^{-|\Omega_{\infty}|^2/\omega^2}$$

for the probability of finding the oscillator in the ground state $|0\rangle$ to $\Omega = 0$, when looking for it at time T.

(b) We have $|1\rangle = A^+|0\rangle$, $\langle 1| = \langle 0|A$, so that

$$\text{prob}(\text{in } |1\rangle \text{ at } T) = \langle 0|A\,\rho(T)\,A^+|0\rangle$$

where $A\,\rho(T)\,A^+ = AA^+\rho(T) + A[\rho(T), A^+]$

$$= AA^+\rho(T) + A\frac{\partial}{\partial A}\rho(T)$$

$$= AA^+\rho(T) - A(A^+ - \Omega_\infty^*/\omega)\,\rho(T)$$

$$= (\Omega_\infty^*/\omega)\,A\,\rho(T)$$

$$= (\Omega_\infty^*/\omega)\left(\rho(T)A + [A, \rho(T)]\right)$$

$$= (\Omega_\infty^*/\omega)\left(\rho(T)A + \frac{\partial}{\partial A^+}\rho(T)\right)$$

$$= (\Omega_\infty^*/\omega)\left(\rho(T)A - \rho(T)(A - \Omega_\infty/\omega)\right)$$

$$= |\Omega_\infty/\omega|^2\,\rho(T),$$

so that $\langle 0|A\,\rho(T)\,A^+|0\rangle$

$$= |\Omega_\infty/\omega|^2 \langle 0|\rho(T)|0\rangle$$

$$= |\Omega_\infty/\omega|^2\, e^{-|\Omega_\infty/\omega|^2}.$$

Note: There are quite a few different methods for getting this result, such as making use of the explicit Gaussian wave functions.

$\boxed{3}$ We have $V_2(\vec{R}) = e^{-i\vec{a}\cdot\vec{P}/\hbar} V_1(\vec{R}) e^{i\vec{a}\cdot\vec{P}/\hbar}$

$$\equiv W^+ V_1(\vec{R}) W$$

with the unitary W commuting with $H_0 = \frac{1}{2M}\vec{P}^2$ and therefore also with $G = (E - H_0 + i\epsilon)^{-1}$. In the equation for the transition operator to V_2,

$$T_2 = V_2 + V_2 G T_2 ,$$

we have then

$$T_2 = W^+ V_1 W + W^+ V_1 G W T_2$$

or

$$W T_2 W^+ = V_1 + V_1 G\, W T_2 W^+ ,$$

so that $W T_2 W^+$ obeys the equation for the transition operator to V_1:

$$T_1 = V_1 + V_1 G T_1 .$$

It follows that $W T_2 W^+ = T_1$, or

$$T_2 = W^+ T_1 W .$$

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$\boxed{4}$ (a) In $T = V + VGV + VGVGV + \cdots$, every term has one V to the left and one V to the right, so that $T|\rangle$ is a linear combination of $|s_1\rangle$ and $|s_2\rangle$, and $\langle|T$ is a superposition of $\langle s_1|$ and $\langle s_2|$. Therefore,

$$T = \left(|s_1\rangle, |s_2\rangle\right) \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \langle s_1| \\ \langle s_2| \end{pmatrix}.$$

(b) Taking matrix elements of $T = V + VGT$ we get

$$T_{11} = \langle s_1|T|s_1\rangle = V_1 + V_1 \langle s_1|GT|s_1\rangle$$

$$= V_1 + V_1 \langle s_1|G\left(|s_1\rangle T_{11} + |s_2\rangle T_{21}\right)$$

$$= V_1 + V_1\left(G_{11}T_{11} + G_{12}T_{21}\right)$$

with $G_{11} = \langle s_1|G|s_1\rangle$, $G_{12} = \langle s_1|G|s_2\rangle$. Likewise

$$T_{21} = V_2 G_{21} T_{11} + V_2 G_{22} T_{21},$$

$$T_{12} = V_1 G_{11} T_{12} + V_1 G_{12} T_{22},$$

$$T_{22} = V_2 + V_2 G_{21} T_{12} + V_2 G_{22} T_{22}.$$

We solve for T_{11}, T_{12}, T_{21}, and T_{22} and get

Question 5/6

Write answers on this side of the paper only.

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$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \right]^{-1} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

$$= \frac{1}{DET} \begin{pmatrix} V_1 - V_1 V_2 G_{22} & V_1 G_{12} V_2 \\ V_2 G_{21} V_1 & V_2 - V_2 V_1 G_{11} \end{pmatrix}$$

with $DET = (1 - V_1 G_{11})(1 - V_2 G_{22}) - V_1 G_{12} V_2 G_{21}$.

$\boxed{5}$ When composing $j=2$ and $j=1$ from $j=3/2$ and $j=1/2$, we have $|j=m=2\rangle = |\tfrac{3}{2};\tfrac{1}{2}\rangle$

short for $j=\tfrac{3}{2}, m=\tfrac{3}{2}$

short for $j=\tfrac{1}{2}, m=\tfrac{1}{2}$

so that $|j=2, m=1\rangle \propto J_- |\tfrac{3}{2};\tfrac{1}{2}\rangle$

$$= |\tfrac{1}{2};\tfrac{1}{2}\rangle \hbar \sqrt{(\tfrac{3}{2}+\tfrac{3}{2})(\tfrac{3}{2}-\tfrac{3}{2}+1)}$$

$$+ |\tfrac{3}{2};-\tfrac{1}{2}\rangle \hbar \sqrt{(\tfrac{1}{2}+\tfrac{1}{2})(\tfrac{1}{2}-\tfrac{1}{2}+1)}$$

$$= |\tfrac{1}{2};\tfrac{1}{2}\rangle \hbar\sqrt{3} + |\tfrac{3}{2};-\tfrac{1}{2}\rangle \hbar .$$

The state with $j=1, m=1$ is orthogonal to this, but a superposition of the same two kets. Properly normalized, we have

$$|j=1, m=1\rangle = \left(|\tfrac{3}{2};-\tfrac{1}{2}\rangle\sqrt{3} - |\tfrac{1}{2};\tfrac{1}{2}\rangle\right)/2 .$$

Another application of J_- gives

$$J_- |j=1, m=1\rangle = |j=1, m=0\rangle \hbar \sqrt{(1+1)(1-1+1)}$$

$$= \left(J_- |\tfrac{3}{2}; -\tfrac{1}{2}\rangle \sqrt{3} - J_- |\tfrac{1}{2}; \tfrac{1}{2}\rangle\right)/2$$

$$= \left(|\tfrac{1}{2}; -\tfrac{1}{2}\rangle \hbar \sqrt{3}^2 - |-\tfrac{1}{2}; \tfrac{1}{2}\rangle \hbar \sqrt{4} \right.$$

$$\left. - |\tfrac{1}{2}; -\tfrac{1}{2}\rangle \hbar \right)/2$$

$$= \left(|\tfrac{1}{2}; -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}; \tfrac{1}{2}\rangle\right) \hbar \, ,$$

so that $|j=1, m=1\rangle = \left(|\tfrac{1}{2}; -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}; \tfrac{1}{2}\rangle\right)/\sqrt{2}$

m_1 m_2.

(a) These probabilities are

0 for $m_2 = \frac{3}{2}$ and $m_2 = -\frac{3}{2}$,

$\frac{1}{2}$ for $m_2 = \frac{1}{2}$ and $m_2 = -\frac{1}{2}$.

(b) These probabilities are $\frac{1}{2}$ each.

(c) This probability is $\frac{1}{2}$.