1. Commutation Relations (10 marks)

Express the commutator $[A, B^{-1}]$ in terms of $[A, B]$, whereby operator A is arbitrary and operator B is invertible.

2. Time-dependent Perturbation (25=15+10 marks)

The Hamilton operator

$$H = \hbar\omega A^\dagger A - \hbar\Omega(t)^* A - \hbar\Omega(t) A^\dagger$$

describes a harmonic oscillator (circular frequency ω, ladder operators $A^\dagger, A$) that is driven by the time-dependent coupling $\hbar\Omega(t)$ which increases *slowly* from its initial value $\Omega(t < 0) = 0$ to its final value $\Omega(t > T) = \Omega_\infty$. Before the drive is applied, the oscillator is in its ground state.

(a) What is the probability of finding the oscillator at time $t = T$ in this ground state, namely the ground state to $\Omega = 0$?

(b) What is the probability of finding the oscillator at time $t = T$ in the first excited state to $\Omega = 0$?

Hint: It may be helpful to recall the normally ordered form of the projector to the oscillator ground state.

3. Scattering (20 marks)

If $\dfrac{d\sigma_1}{d\Omega}$ is the differential scattering cross section to the localized scattering potential $V_1(\vec{R})$, what is the differential scattering cross section $\dfrac{d\sigma_2}{d\Omega}$ to the scattering potential $V_2(\vec{R}) = V_1(\vec{R} - \vec{a})$, where $\vec{a}$ is a numerical displacement vector? Justify your answer.

Note: Full marks will be earned only by answers that do not invoke an approximation, such as the Born approximation.

4. Scattering (20=8+12 marks)

The scattering potential in $H = \dfrac{1}{2M}\vec{P}^{\,2} + V$ is the sum of two separable potentials,

$$V = |s_1\rangle V_1 \langle s_1| + |s_2\rangle V_2 \langle s_2| \,,$$

where V_1 and V_2 are real parameters and the kets $|s_1\rangle$, $|s_2\rangle$ are orthonormal: $\langle s_1|s_1\rangle = \langle s_2|s_2\rangle = 1$ and $\langle s_1|s_2\rangle = 0$.

(a) Explain why the transition operator has the form

$$T = |s_1\rangle T_{11} \langle s_1| + |s_1\rangle T_{12} \langle s_2| + |s_2\rangle T_{21} \langle s_1| + |s_2\rangle T_{22} \langle s_2| \,.$$

(b) Determine the values of T_{11}, T_{12}, T_{21}, T_{22}.

5. Angular Momentum (25=10+10+5 marks)

An unstable molecule is in the angular momentum state with $j = 1$ and $m = 0$ and decays spontaneously into two fragments with $j_1 = \frac{3}{2}$ and $j_2 = \frac{1}{2}$.

(a) What are the probabilities of finding the first fragment with $m_1 = \frac{3}{2}$, $m_1 = \frac{1}{2}$, $m_1 = -\frac{1}{2}$, or $m_1 = -\frac{3}{2}$, respectively?

(b) What are the probabilities of finding the second fragment with $m_2 = \frac{1}{2}$ or $m_2 = -\frac{1}{2}$, respectively?

(c) What is the joint probability of finding the first fragment with $m_1 = \frac{1}{2}$ and the second fragment with $m_2 = -\frac{1}{2}$?