Question. . . 1/3 . . .

Write answers on this side of the paper only.

Do not write on either margin

[1] (a) $f(-a) = f(a)^*$.

(b) $f(a) = \langle 1 | \underbrace{\left(\int dx\, |x-a\rangle\langle x+a| \right)}_{W_a(X,P)} |1\rangle$, so that

$$W_a(X,P) = e^{iaP/\hbar} \underbrace{\int dx |x\rangle\langle x|}_{=1}\, e^{iaP/\hbar}$$

$$= \underline{e^{2iaP/\hbar}}.$$

(c) Since $f(a) = \langle e^{2iaP/\hbar} \rangle$, we have $f(0) = 1$,

$$\frac{\partial}{\partial a} f(a) \Big|_{a=0} = \frac{2i}{\hbar} \langle P \rangle, \quad \left(\frac{\partial}{\partial a}\right)^2 f(a) \Big|_{a=0} = \left(\frac{2i}{\hbar}\right)^2 \langle P^2 \rangle,$$

and so forth. Here,

$$f(a) = \left(\frac{1}{1+(ka)^2} \right)^2 = 1 - 2(ka)^2 + O\big((ka)^4\big),$$

giving $\langle P \rangle = 0$, $\langle P^2 \rangle = (\hbar k)^2$, and then

$$\underline{\underline{\delta P = \hbar k.}}$$

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[2] With $\langle x | Z | p \rangle = e^{ixp/\hbar} \langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{2ixp/\hbar}$

we have

$$\frac{\langle p|Z|x\rangle}{\langle p|x\rangle} = \int dx' dp' \frac{\langle p|x'\rangle\langle x'|Z|p'\rangle\langle p'|x\rangle}{\langle p|x\rangle}$$

$$= \int \frac{dx'dp'}{2\pi\hbar} e^{-ipx'/\hbar} e^{2ix'p'/\hbar} e^{-ip'x/\hbar} e^{ipx/\hbar}.$$

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$$= \int dx'\, e^{-ipx'/\hbar} e^{ipx/\hbar}\, \delta(2x'-x)$$

$$= \tfrac{1}{2}\, e^{i\frac{1}{2}px/\hbar},$$

so that $Z = \frac{1}{2}\, e^{\frac{i}{2} P; X/\hbar}$

is the P, X - ordered form of Z.

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[3] $LHS = e^{ix'P/\hbar} e^{-ip'X/\hbar} e^{-ip''X/\hbar} e^{ix''P/\hbar} \cdot e^{\frac{i}{2}(x'p'-x''p'')}$ (1.8.17) used twice

$$= e^{ix'P/\hbar} e^{ix''(P+p'+p'')/\hbar} e^{-i(p'+p'')X/\hbar} \cdot e^{\frac{i}{2}(x'p'-x''p'')/\hbar}$$ (1.8.14) together with (1.8.5)

$$= e^{i(x'+x'')P/\hbar} e^{-i(p'+p'')X/\hbar} e^{ix''p'/\hbar} \cdot e^{\frac{i}{2}(x'p'+x''p'')/\hbar}$$

$$= e^{i((x'+x'')P-(p'+p'')X)/\hbar} e^{-\frac{i}{2}(x'+x'')(p'+p'')/\hbar} \cdot e^{ix''p'/\hbar} e^{\frac{i}{2}(x'p'+x''p'')/\hbar}$$ (1.8.17) once more

$$= e^{i(xP-pX)/\hbar}\, e^{i\Phi}$$

with $x = x' + x''$, $p = p' + p''$,

and $\Phi = \frac{1}{2}(p'x'' - x'p'')/\hbar$.

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[4]

$$\mathrm{tr}\{X \tfrac{\partial\rho}{\partial x}\} = \mathrm{tr}\{X \tfrac{1}{i\hbar}[\rho, P]\} = \tfrac{1}{i\hbar}\,\mathrm{tr}\{X\rho P - XP\rho\}$$

$$= \frac{1}{i\hbar} \mathrm{tr}\{\underbrace{(PX - XP)}_{=-i\hbar}\rho\} = -\mathrm{tr}\{\rho\} = -1.$$

Similarly, $\mathrm{tr}\{P \frac{\partial \rho}{\partial P}\} = -1.$

Note that, more generally, we have

$$\mathrm{tr}\{A[B,C]\} = \mathrm{tr}\{[A,B]C\}$$
$$= \mathrm{tr}\{B[C,A]\}.$$

$\boxed{5}$ (a) $\frac{d}{dt} X = \frac{\partial H}{\partial P} = 2\gamma X, \quad \frac{d}{dt} P = -\frac{\partial H}{\partial X} = -2\gamma P$

are solved by

$$X(t) = X(t_0)\, e^{2\gamma(t-t_0)},$$
$$P(t) = P(t_0)\, e^{-2\gamma(t-t_0)}.$$

(b) Clearly $\frac{d}{dt} F = 0$. To determine $\frac{\partial}{\partial t} F$, we first write F in terms of $X(t)$ and $P(t)$,

$$F = X(t) e^{-2\gamma(t-t_1)} P(t) e^{2\gamma(t-t_2)} X(t) e^{-2\gamma(t-t_1)}$$
$$= e^{-4\gamma(t-t_1)+2\gamma(t-t_2)} X(t)P(t)X(t),$$

so that

$$\frac{\partial}{\partial t} F = -2\gamma F.$$