Problem 1 (5+15+10=30 marks)
The system is in the state described by the position wave function $\psi(x) = \langle x| \,\rangle$. We consider the function $f(a)$ of the real length parameter a that is defined by

$$f(a) = \int_{-\infty}^{\infty} \mathrm{d}x\ \psi(x-a)^*\,\psi(x+a)\,,$$

the so-called "auto-correlation function."

(a) How is $f(-a)$ related to $f(a)$?

(b) Find the operator $W_a(X,P)$, for which $f(a) = \langle\ |W_a(X,P)|\ \rangle$. State $W_a(X,P)$ explicitly as a function of position operator X and momentum operator P.

(c) What is the momentum spread δP if $f(a) = [1+(ka)^2]^{-2}$ with $k>0$?

Problem 2 (20 marks)
The X,P-ordered form of operator $Z = \mathrm{e}^{\mathrm{i}X;P/\hbar}$ is an ordered exponential function. What is the P,X-ordered form of Z?

Problem 3 (20 marks)
For given x', p', x'', and p'', determine x, p, and ϕ such that

$$\mathrm{e}^{\mathrm{i}(x'P-p'X)/\hbar}\,\mathrm{e}^{\mathrm{i}(x''P-p''X)/\hbar} = \mathrm{e}^{\mathrm{i}(xP-pX)/\hbar}\,\mathrm{e}^{\mathrm{i}\phi}$$

holds.

Problem 4 (10 marks)
The state of a particle moving along the x axis is specified by the statistical operator $\rho(X,P)$. Evaluate

$$\mathrm{tr}\left\{X\frac{\partial\rho}{\partial X}\right\} \quad \text{and} \quad \mathrm{tr}\left\{P\frac{\partial\rho}{\partial P}\right\}.$$

Hint: The answers do not depend on the particular choice for $\rho(X,P)$.

Problem 5 (10+10=20 marks)
Consider motion along the x axis with the dynamics governed by the Hamilton operator $H = \gamma(XP+PX)$ with $\gamma > 0$.

(a) State and solve Heisenberg's equations of motion for $X(t)$ and $P(t)$.

(b) Then determine the total time derivative $\frac{\mathrm{d}}{\mathrm{d}t}F$ and the parametric time derivative $\frac{\partial}{\partial t}F$ of $F = X(t_1)P(t_2)X(t_1)$, where t_1 and t_2 are two arbitrary instants.