AY 2006/07, Sem 2, PC3130, Test 2

$\boxed{1}$ (a) $\frac{d}{dt} A = -i\omega(A-\alpha)$, $\frac{d}{dt} A^{+} = i\omega(A^{+}-\alpha^{*})$;

$A(t)-\alpha = e^{-i\omega(t-t_0)}(A(t_0)-\alpha)$ or

$A(t) = \alpha + e^{-i\phi}(A(t_0)-\alpha)$ with $\phi = \omega(t-t_0)$,

$A^{+}(t) = \alpha^{*} + e^{+i\phi}(A^{+}(t_0)-\alpha^{*})$.

(b) $i\hbar \frac{\partial}{\partial t} \langle a^{*},t|a',t_0\rangle = \langle a^{*},t|\, H\, |a',t_0\rangle$

with $H = \hbar\omega(A^{+}(t)-\alpha^{*})(A(t)-\alpha)$

$= \hbar\omega(A^{+}(t)-\alpha^{*})e^{-i\phi}(A(t_0)-\alpha)$

and $\frac{\partial}{\partial t} = \omega\frac{\partial}{\partial\phi}$, so that

$i\frac{\partial}{\partial\phi}\langle a^{*},t|a',t_0\rangle = (a^{*}-\alpha^{*})e^{-i\phi}(a'-\alpha)\langle a^{*},t|a',t_0\rangle$

$= \langle a^{*},t|a',t_0\rangle\, i\frac{\partial}{\partial\phi}\left((a^{*}-\alpha^{*})e^{-i\phi}(a'-\alpha)\right)$

giving

$$\langle a^{*},t|a',t_0\rangle = e^{(a^{*}-\alpha^{*})(e^{-i\phi}-1)(a'-\alpha)}\, e^{a^{*}a'}$$

where $\langle a^{*},t|a',t_0\rangle\Big|_{t\to t_0} = e^{a^{*}a'}$ is

incorporated.

(c) Since $|n=0\rangle = |a'=0\rangle$, $\langle n=0| = \langle a^{*}=0|$ we have

$$\text{probability} = |\langle 0,t|0,t_0\rangle|^2 = \left|e^{\alpha^*(e^{-i\phi}-1)\alpha}\right|^2$$
$$= e^{-2(1-\cos\phi)\alpha^*\alpha} = e^{-|2\alpha\sin\frac{\phi}{2}|^2}.$$

$\boxed{2}$ $$\vec{F}\times\vec{F} = p^2\underbrace{\vec{R}\times\vec{R}}_{=0} + x^2\underbrace{\vec{P}\times\vec{P}}_{=0} + \underbrace{\vec{L}\times\vec{L}}_{=i\hbar\vec{L}}$$
$$+xp\underbrace{(\vec{R}\times\vec{P}+\vec{P}\times\vec{R})}_{=0} + p\underbrace{(\vec{R}\times\vec{L}+\vec{L}\times\vec{R})}_{=2i\hbar\vec{R}}$$
$$+x\underbrace{(\vec{P}\times\vec{L}+\vec{L}\times\vec{P})}_{=2i\hbar\vec{P}},$$ so that
$$\vec{F}\times\vec{F} = i\hbar(2p\vec{R}+2x\vec{P}+\vec{L}) = i\hbar(2\vec{F}-\vec{L}).$$

$\boxed{3}$ (a) $2x^2+y^2=1$.

(b) Since $L_+|\rangle = |2,1\rangle y\hbar\sqrt{6} + |2,-1\rangle x\hbar\sqrt{4}$,
$$L_-|\rangle = |2,1\rangle x\hbar\sqrt{4} + |2,-1\rangle y\hbar\sqrt{6}$$
we have
$$L_1|\rangle = \frac{1}{2}(L_++L_-)|\rangle = (|2,1\rangle+|2,-1\rangle)\hbar(x+y\sqrt{3/2}),$$
$$L_2|\rangle = \frac{1}{2i}(L_+-L_-)|\rangle = (|2,1\rangle-|2,-1\rangle)i\hbar(x-y\sqrt{3/2}),$$
as well as
$$L_3|\rangle = (|2,2\rangle-|2,-2\rangle)2\hbar x.$$
Therefore, $\langle L_1\rangle = \langle L_2\rangle = \langle L_3\rangle = 0$,
$$\langle L_1^2\rangle = \hbar^2(\sqrt{2}x+\sqrt{3}y)^2,$$
$$\langle L_2^2\rangle = \hbar^2(\sqrt{2}x-\sqrt{3}y)^2,$$

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and $\delta L_1 = \hbar \, |\sqrt{2}\, x + \sqrt{3}\, y|$,

$\delta L_2 = \hbar \, |\sqrt{2}\, x - \sqrt{3}\, y|$,

$\delta L_3 = \hbar \sqrt{8}\, |x|$ follow.

$\boxed{4}$ $\frac{d}{dt} H = 0$: $\frac{1}{2M}\left(\vec{P}\cdot\frac{d\vec{P}}{dt} + \frac{d\vec{P}}{dt}\cdot\vec{P}\right) = Ze^2 \frac{d}{dt}\frac{1}{|\vec{R}|}$

and with

$$\frac{1}{M}\vec{P} = \frac{d}{dt}\vec{R}, \quad \frac{d}{dt}\vec{P} = -Ze^2 \frac{\vec{R}}{|\vec{R}|^3}$$

this gives

$$\frac{d}{dt}\frac{1}{|\vec{R}|} = -\frac{1}{2}\left(\frac{d\vec{R}}{dt}\cdot\frac{\vec{R}}{|\vec{R}|^3} + \frac{\vec{R}}{|\vec{R}|^3}\cdot\frac{d\vec{R}}{dt}\right).$$

$\boxed{5}$ $\left.\frac{d}{d\lambda} E_n(\lambda)\right|_{\lambda=0} = \langle n| \left.\frac{\partial H}{\partial\lambda}\right|_{\lambda=0} |n\rangle = \langle n|D^\dagger D|n\rangle$

with

$$D|n\rangle = \begin{cases} |n\rangle + |n-1\rangle & \text{for } n = 1, 2, 3, \ldots \\ |0\rangle & \text{for } n = 0 \end{cases}$$

so that

$$\left.\frac{d}{d\lambda} E_n(\lambda)\right|_{\lambda=0} = \begin{cases} 2 & \text{for } n = 1, 2, 3, \ldots, \\ 1 & \text{for } n = 0. \end{cases}$$